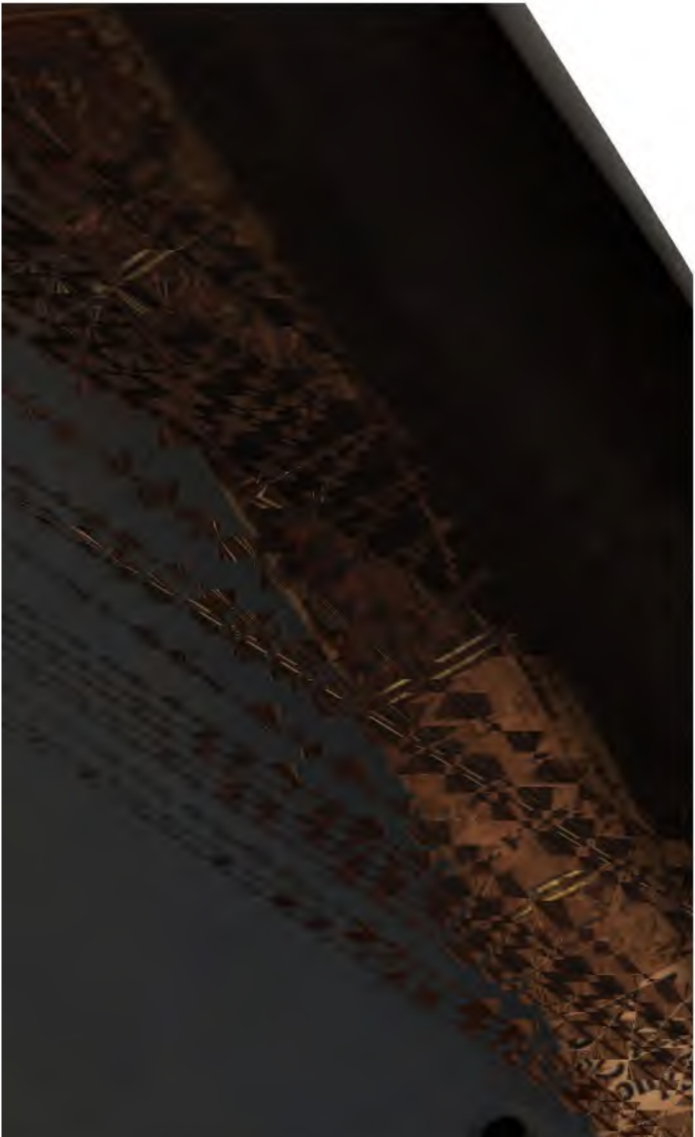

ELEMENTS

OF

ARITHMETIC,

FOR

THE USE OF SCHOOLS.

BY

NOBLE HEATH.

NEW-YORK:

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1826.

Southern District of New-York, ss.

BE IT REMEMBERED, that on the 17th day of February, in the fiftieth year of the Independence of the United States of America, A. D. 1826, Noble Heath, of the said District, has deposited in this office the title of a book, the right whereof he claims as Author, in the words following, to wit:

"Elements of Arithmetic, for the Use of Schools. By Noble Heath."

In conformity to the Act of Congress of the United States, entitled, "An Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned." And also to an Act, entitled, "An Act, supplementary to an Act, entitled an Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned, and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

JAMES DILL,

Clerk of the Southern District of New-York.

PREFACE.

To add one more to the number of elementary treatises on arithmetic would seem, in the abstract, to be a useless task; but every person who has become duly sensible, by long experience in teaching, of the importance of inculcating on the minds of children, in the commencement of their exercises, clear and intelligible ideas, and a just perception of the reason of the rules that are given them, will not hesitate to admit, that a work which adds to the facilities of the teacher or learner, may have a just claim upon those whose important duty it is to conduct youth through the paths of knowledge. In examining the arithmetic of the celebrated French writer Bezout, the author was so impressed with the superiority of its arrangement and elucidations over any known English or American book, that he translated and published that work for the benefit of American schools. It is now before the public, and has met with ready acceptance on the part of many distinguished teachers. It has been suggested, however, by various persons, as well as by the author's own ex-

perience, that a book less extensive than Bezout, for the use of scholars in the early stages of learning, with easy and natural explanations, a variety of examples for practice, and verbal questions for the pupil to answer, embracing the necessary rules and principles, would form a valuable school book. Such is the design of the present work. Should the scholar who has been conducted through it be found to have a clearer understanding of the primary rules, and of fractional as well as integral combinations, than is generally the case with pupils at the same period of their progress, the anticipations of the author will be realized, not only in the saving of time, but in the pleasure which the scholar will derive in advancing into the higher parts of arithmetic on a solid foundation.

The division of the subject into sections, with questions at the end of each calculated to elicit an explanation of principles, will, it is presumed, adapt the work to the use of schools conducted on the monitorial system, as well as render it more useful to others.

Trusting that enlightened teachers will clearly perceive that arithmetic thus taught will be an intelligible and agreeable study, the work is submitted with confidence to their candid judgment.

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ARITHMETIC.

Section 1.

PRELIMINARY IDEAS.

1. That of which we can conceive more and less, or which may be measured, is called *quantity*. Thus any portion of time, of space, of weight, etc. is a quantity.

2. To measure a quantity, we take some known quantity of the same kind as a standard with which it may be compared. For example, to convey an idea of the distance between two places, we compare this distance to a foot, a yard, a mile, a league, or any other known measure of length.

3. The quantity or measure which we use in comparison is called *a unit*.

4. Number shows how many units a quantity contains.

5. Arithmetic is the science of numbers: it explains their nature and properties, and shows their application to the various purposes of life.

QUESTIONS ON SECTION 1.

1. What is quantity?
2. How do we measure a quantity?
3. What is the quantity called which we use in comparision?
4. What does number show?
5. What is arithmetic, and what does it explain?

SECTION 2.

NUMERATION.

6. Numeration is the art of expressing numbers. Numbers are formed by the repeated addition of a single unit, and are all represented by the following ten characters or figures.

nought,	one,	two,	three,	four,	five,	six,	seven,	eight,	nine,
0,	1,	2,	3,	4,	5,	6,	7,	8,	9,

The character 0 never represents any number; the other nine are, by way of distinction, called *significant figures*, and sometimes also *digits*:

NOTE. As it is from the fingers that we are supposed to derive our numeration, the nine significant figures are called digits, from the Latin word *digitus*, a finger.

The figure 1 represents a single unit; the figure 2, a unit added to a unit; the figure 3, a unit added to 2, and thus we may procede to an infinity of numbers, because a number can never be so great but that we may still add a unit to it.

7. Having proceeded as far as nine, we find ourselves at a stand, as we have no single character with which we can represent the next number, which is formed by the addition of 1 to 9.

This next number is called *ten*, and is represented by the single unit 1, placed on the left of the cipher, thus 10.

Now this single unit 1 does not derive its present value of 10 units from the cipher, which does not signify any number; but the cipher now occupies the place which the 1 would have occupied if it had stood alone, and this 1 derives its present value of *ten units* from its having been removed one place farther towards the left. For this reason, when two figures stand by the side of each other, the figure on

the right hand is said to stand in the place of units, and that on the left in the place of tens.

8. Having expressed ten units by a single unit, removed one place towards the left, the place of units is left open to receive the units in succession from 1 to 9 inclusive; and thus the next number *eleven*, which is formed by the addition of 1 to 10, is expressed by putting the unit 1 in its proper place, and is consequently written thus 11.

Continuing to add a unit we form the numbers 12, 13, 14, 15, 16, 17, 18, 19, which are called *twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen*, and which are ten and two, ten and three, ten and four, ten and five, ten and six, ten and seven, ten and eight, and ten and nine. Hence we see that the figure in the place of units has no effect upon the *ten*, which remains the same in all these numbers.

9. The next number in succession is formed by adding 1 to 19. Now in adding 1 to the 9 units of this number, we have ten units; but ten units are expressed by a single unit put in the place of tens, (art. 7:) we therefore add a unit to the unit already in that place, which gives 2, or 2 tens, and lest this 2 should be taken for 2 simple units, we again have recourse to the cipher, and write it thus 20, and instead of saying *two tens*, we say *twenty*.*

10. The place of units having again become open, we replace them as before, in doing which we form the numbers 21, 22, 23, 24, 25, 26, 27, 28, 29, which we read *twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine*, and which are two tens and one, two tens and two, two tens and three, etc.

11. To form the next number in succession, as 29 signifies 2 tens and 9 units, in adding a unit to this number, (articles 7 and 9,) we have 3 tens, which we call *thirty*, and write thus 30. Continuing the addi-

* Twain ten.

tion of a unit, we form the numbers 40, 50, 60, 70, 80, 90, which are read *forty*, *fifty*, *sixty*, *seventy*, *eighty*, *ninety*, and which signify *four tens*, *five tens*, *six tens*, *seven tens*, *eight tens*, and *nine tens*. Again replacing the units, we form the numbers 91, 92, 93, 94, 95, 96, 97, 98, 99, which are read *ninety-one*, *ninety-two*, etc. to *ninety-nine*.

12. To form the next number we add 1 to 99. Now in adding 1 to the 9 units we have ten, and ten is expressed by putting the figure 1 in the place of tens: this 1 together with the 9 which is already in the place of tens, gives ten tens, and as we have no character greater than 9, we express these ten tens in the same manner that we expressed ten single units: that is to say, by putting the same figure 1 on the left of the place of tens: and that this figure 1 may not be taken for a single unit nor a ten, we write it on the left of two ciphers thus 100, and read *one hundred*.

13. The place of units and the place of tens being now open, we might procede to replace them in the same manner, and thus we should form all the numbers from 101, *one hundred and one*, to 199, *one hundred and ninety-nine*, inclusive.

14. As 1 added to 99 gives 100, (art. 12,) we easily see that the next number, which is formed of the addition of 1 to 199, will be 200, which is read *two hundred*, and that by continuing to replace the units and the tens, we might form the numbers 300, 400, 500, 600, 700, 800, 900, which are read *three hundred*, *four hundred*, *five hundred*, *six hundred*, *seven hundred*, *eight hundred*, and *nine hundred*. Because each unit in the third place towards the left signifies *one hundred*, this place is called the place of hundreds.

15. From 900, by replacing the units and tens, we can procede as far as 999, which is the largest number that can be represented by three places of figures, because we have no figure greater than 9, and consequently, to express the next number, which

is formed by adding 1 to 999, we occupy four places; that is to say, as 1 added to 99 gives 100, (art. 12,) and as the addition of 1 hundred to 9 hundreds gives 10 hundreds, we express these ten hundreds by putting a single unit on the left of the place of hundreds, thus 1000, which number is called *one thousand*. Thus we see that by continuing to express ten units of any order by a single unit placed on the left, we might procede in the formation of numbers without end. Hence also we see that the units composing any figure in any place towards the left hand, are ten times as great as they would be if the figure stood one place farther towards the right.

16. In arithmetical calculations, large numbers frequently occur, consisting of many more places of figures than the numbers above enumerated: and though we seldom give ourselves the trouble of reading those numbers, yet it is important to be able to do this with facility. For this purpose, we separate the proposed number into periods of three figures each, beginning at the right hand and proceding towards the left; these periods we call by the following names, beginning at the right, *units*, *thousands*, *millions*, *billions*, *trillions*, *quadrillions*, *quintillions*, *sextillions*, *septillions*, *octillions*, *nonillions*, *decillions*, *undecillions*, *duodecillions*, *tredecillions*, *quatuordecillions*, *quindecillions*, *sexdecillions*, *septendecillions*, *octodecillions*, etc. The figures of each period are read precisely in the same manner as if they stood alone, except merely pronouncing the name of the period after reading them. For example, the following number is read, beginning at the left hand,

quadrillions,	trillions,	billions,	millions,	thousands,	units,
56	111	317	204	869	020

fifty-six *quadrillions*, one hundred and eleven *trillions*, three hundred and seventeen *billions*, two hundred and four *millions*, eight hundred and sixty-nine *thousand* and twenty.

In separating numbers into periods of three figures each, the last or left hand period will frequently contain only one or two figures; thus, in the above example, the 5 which is fifty quadrillions might be taken away, and then we should have 6 units under that name, that is, 6 quadrillions. But as it is to periods of three figures each that we give the names above written, we could not constitute another period on the left of 56 quadrillions without occupying the place of hundreds, which is now vacant by a cipher: there are therefore three places of figures, that is to say, the place of *units*, the place of *tens*, and the place of *hundreds*, under each name, which places must either be occupied by figures or ciphers under every period except the left-hand period, and even here the cipher cannot be omitted except when it would stand on the left of all the significant figures.

17. We do not pronounce the name units after reading the last period, because all numbers are composed of units, and therefore this term is as applicable to all the other periods as to the last. For example, we have seen in the formation of numbers (articles 7, 12, and 15) that the place of *units*, that of *tens*, and that of *hundreds*, which constitute this period, have each successively become occupied by a cipher only, we might therefore have such a number as this, viz. 657000: now after having separated this number into periods of three figures each, thus,

thousands,	units,
657	000

we read, six hundred and fifty-seven *thousand*. But this is a number;—a number of what? of units.

18. When a large number is expressed in words, in order to write down its appropriate figures, we first write, in their proper order, (art. 16,) the names of the different periods, beginning with the units and terminating with the highest name mentioned in the *number*; after which, as there are three places of *figures* under each name, (art. 16,) we put such

figures in these places as the reading of the number dictates, beginning with the left-hand period or highest name. For example, to express in figures the following number, *sixty quintillions, fifteen quadrillions, one hundred and one trillions, ten millions, one hundred thousand and ten*, we first write the names of the different periods thus:—

quintillions,	quadrillions,	trillions,	billions,	millions,	thousands,	units,
60	015	101	000	010	100	010

then as the proposed number first dictates *sixty quintillions*, we write 60 under quintillions, putting a cipher in the place of units, and leaving the place of hundreds vacant; then as the number requires *fifteen quadrillions*, we write 15 under quadrillions, but we must not here leave the place of hundreds vacant, by omitting the cipher, as we did under quintillions; for, as the 6 which stands under quintillions derives its present value of sixty quintillions from the number of places to which it is removed towards the left, that omission would, by bringing it back again one place towards the right, make it signify only *six quintillions*, that is to say, (art. 15,) only one tenth part of its intended value; we therefore write 015 under quadrillions; this done, as the proposed number next dictates *one hundred and one trillions*, we write 101 under trillions: again, for the *billions*;— but there are none; we therefore write 000 under billions, for, if we omitted these, the trillions would be billions; the quadrillions would become trillions, the quintillions quadrillions, etc. Now, for the *millions*, as the number dictates *ten*, and as ten is signified by putting 1 in the place of tens, (art. 7,) we write 010, putting ciphers in the other places for the reasons given above. Again, for the *thousands*, as the number dictates *one hundred*, and as this is expressed by putting a single unit in the place of hundreds, (art. 12,) we write 100. Lastly, as the number dictates *ten*, we write this ten under units in the

same manner that we wrote ten under millions, thus 010, which completes the proposed number.

Here, remark that the cipher, though of no value in itself, is of great importance, as it occupies the place of any order of units which is wanting in a number, and consequently serves to determine the value of the significant figures by showing their position towards the left.

19. From what has been said, we see that in order to read and write numbers with ease and accuracy, no other knowledge is required than to be able to read and write correctly any number consisting of three places of figures, and to remember the names and order of the different periods.*

Let the following numbers be expressed in words.

1. 10056.	4. 230110012.
2. 116400.	5. 913215010.
3. 19130070.	6. 12030112904.

7. 230506278591000063002.
8. 1327100453806017134520l6.
9. 720381032110006101235083200064.
10. 300291330190340000060708103021811117004.

Let the following numbers be expressed in figures.

1. Twenty thousand and two.
2. Fifteen millions and sixty.
3. Four hundred billions, thirty thousand and four.
4. Six hundred and three quadrillions, fifty-six billions and sixteen.

* Most English arithmeticians apply the names *millions, billions,* etc. to periods of six figures, considering each of these periods as being composed of two periods of three figures each, by which means they have a repetition of thousands under each name. Now this repetition of thousands renders the subject of numeration much more difficult to the student, and can be of no possible utility; for a very simple calculation will show that if the whole globe that we inhabit was divided into particles so small that it would require ten millions of them to make a solid inch, which would be of the fine*ness of fine* sand, the whole number of these particles contained in *the globe,* according to the numeration of three figures, would not *equal the* hundredth part of a decillion.

5. Five hundred and eleven septillions, forty-one quintillions, eleven quadrillions, one hundred millions, three hundred thousand and forty.

6. One hundred and six sextillions, four hundred and ninety quadrillions, three billions, seventeen millions, six hundred and four thousand and three.

7. Twenty quintillions, one hundred and eleven trillions, ninety billions, seven thousand three hundred.

8. Forty-nine septillions, three hundred and twenty-seven quintillions, fourteen quadrillions, one hundred and ten trillions, ten thousand and forty.

9. Eight octillions, seventeen septillions, five hundred sextillions, one hundred and sixty thousand and nine.

10. One decillion, five octillions, seven hundred quintillions, ten quadrillions, eleven billions, one hundred and nine millions, four hundred and two thousand one hundred and six.

11. Nine hundred and six octodecillions, seventy-two sexdecillions, eighteen duodecillions, five hundred undecillions, twenty-nine nonillions, one hundred and eleven septillions, sixty quadrillions, five trillions, twelve millions, nine hundred thousand and six.

QUESTIONS ON SECTION 2.

1. What is numeration?

2. How are numbers formed?

3. How many characters are used in representing numbers, and what are they?

4. Why are the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 called significant figures, and why do we sometimes call them digits?

5. How are ten units expressed, and what is the general rule for expressing ten units of any order?

6. Does the unit, when standing on the left of a cipher, derive any additional value from the cipher;

and if not, from what circumstance is it in this case, considered as representing ten units?

7. Does a single unit always signify ten when standing in the second place of figures towards the left?

8. What does the figure 3 signify when standing in that place?

9. What is the separate value of each of the figures which express the number one hundred and thirty-two?

10. By what significant figure do we express ninety, and where must it stand?

11. By which of the digits do we express ten tens, and where must it stand?

12. Why is the third place of figures towards the left called the place of hundreds?

13. How are numbers divided in order to enumerate them?

14. What are the names of the different periods, and how are they arranged?

15. Where do we begin and how do we procede in reading numbers?

16. Does the left hand period of a number always contain three figures?

17. Must every period except that on the left always contain three figures?

18. Of what use is the cipher?

19. Of what use is it when placed on the left of all the significant figures?

20. How do we procede in order to express in figures a large number which is written in words?

21. If through mistake we omit one of the intermediate ciphers in a number, what will be the effect?

22. If we omit any one of the intermediate periods in a number which ought to be supplied with ciphers, what will be the effect?

23. Why do we omit the name of the last period in reading a number?

24. What knowledge is requisite in order to read and write all numbers with facility?

25. Why is the numeration of three figures to a period preferable to that of six?

SECTION 3.

ADDITION.

20. Addition teaches to express, by a single number, all the units contained in two or more given numbers of whatever extent they may be.

21. This single number is called the sum of the given numbers: thus we say that the sum of 4 and 3 is 7, which is a single number containing as many units as are contained in both 4 and 3.

22. A sign made thus +, which is called *plus* or *more*, is placed between numbers to show that they are to be added together; thus the expression 9+1 shows that 9 and 1 are to be added together, and is read nine *plus* one.

Also a sign made thus =, which is called *equal to*, is put between quantities to show that those on the one side of it are equal to those on the other; thus, we write 9+1=10, which expression signifies that the sum of 9 and 1 is equal to 10, and is read nine *plus* one *equal to* ten.

Let each expression in the following examples be added from the left hand towards the right, and also from the right hand towards the left; if the sum is found the same both ways, the work can scarcely be wrong.

EXAMPLES.

1. 2+3+4=9; 4+2+3+3=12; 5+2+3+4 =14.
2. 3+2+4+6+3=18, and 5+3+4+2+6=20.
3. 6+4+3+7+5=25, and 7+5+7+3=22.
4. 8+3+4+7+8+9+6=
5. 5+4+9+7+8+6+5+7+9=
6. 8+8+6+4+5+6+8+9+7+9=

7. $2+3+6+9+5+7+8+9+5+9+8+5=$
8. $9+2+8+9+8+7+4+5+6+5+6+3=$

23. The following examples should be remembered for future purposes.

Ex. 1. $2+2=4$; $3+3=6$; $4+4=8$; $5+5=10$; $6+6=12$; $7+7=14$; $8+8=16$; $9+9=18$. In this example, we may observe that, having added the figure 2 to itself, we have the value of twice 2, and that it is the same with regard to the other figures: we therefore say that twice 2 is 4; twice 3 is 6; twice 4 is 8; twice 5 is 10; twice 6 is 12; twice 7 is 14; twice 8 is 16, and that twice 9 is 18.

Ex. 2. $2+2+2=6$; $3+3+3=9$; $4+4+4=12$; $5+5+5=15$; $6+6+6=18$; $7+7+7=21$; $8+8+8=24$, and $9+9+9=27$. Hence, for the same reason as above, we say, 3 times 2 is 6; 3 times 3 is 9; 3 times 4 is 12; 3 times 5 is 15; 3 times 6 is 18; 3 times 7 is 21; 3 times 8 is 24, and 3 times 9 is 27.

Ex. 3. $2+2+2+2=8$; $3+3+3+3=12$; $4+4+4+4=16$; $5+5+5+5=20$; $6+6+6+6=24$; $7+7+7+7=28$; $8+8+8+8=32$; $9+9+9+9=36$. Hence, we say, 4 times 2 is 8; 4 times 3 is 12; 4 times 4 is 16; 4 times 5 is 20; 4 times 6 is 24; 4 times 7 is 28; 4 times 8 is 32, and 4 times 9 is 36.

Ex. 4. $2+2+2+2+2=10$; $3+3+3+3+3=15$; $4+4+4+4+4=20$; $5+5+5+5+5=25$; $6+6+6+6+6=30$; $7+7+7+7+7=35$; $8+8+8+8+8=40$; and $9+9+9+9+9=45$. Hence, we see that 5 times 2 is 10; 5 times 3 is 15; 5 times 4 is 20; 5 times 5 is 25; 5 times 6 is 30; 5 times 7 is 35; 5 times 8 is 40, and that 5 times 9 is 45.

Ex. 5. $2+2+2+2+2+2=12$; $3+3+3+3+3+3=18$; $4+4+4+4+4+4=24$; $5+5+5+5+5+5=30$; $6+6+6+6+6+6=36$; $7+7+7+7+7+7=42$; $8+8+8+8+8+8=48$, and $9+9+9+9+9+9=54$. Hence, we say, 6 times 2 is 12; 6 times 3 is 18; 6 times 4 is 24; 6 times 5 is 30; 6 times 6 is 36; 6 times 7 is 42; 6 times 8 is 48, and 6 times 9 is 54.

Ex. 6. 2+2+2+2+2+2+2=14; 3+3+3+3+3+3+3=21; 4+4+4+4+4+4+4=28; 5+5+5+5+5+5+5=35; 6+6+6+6+6+6+6=42; 7+7+7+7+7+7+7=49; 8+8+8+8+8+8+8=56, and 9+9+9+9+9+9+9=63. Hence, we say, 7 times 2 is 14; 7 times 3 is 21; 7 times 4 is 28; 7 times 5 is 35; 7 times 6 is 42; 7 times 7 is 49; 7 times 8 is 56, and 7 times 9 is 63.

Ex. 7. 2+2+2+2+2+2+2+2=16; 3+3+3+3+3+3+3+3=24; 4+4+4+4+4+4+4+4=32; 5+5+5+5+5+5+5+5=40; 6+6+6+6+6+6+6+6=48; 7+7+7+7+7+7+7+7=56; 8+8+8+8+8+8+8+8=64, and 9+9+9+9+9+9+9+9=72. In this example, having added eight figures of the same kind together, we say, 8 times 2 is 16; 8 times 3 is 24; 8 times 4 is 32; 8 times 5 is 40; 8 times 6 is 48; 8 times 7 is 56; 8 times 8 is 64, and 8 times 9 is 72.

Ex. 8. 2+2+2+2+2+2+2+2+2=18; 3+3+3+3+3+3+3+3+3=27; 4+4+4+4+4+4+4+4+4=36; 5+5+5+5+5+5+5+5+5=45; 6+6+6+6+6+6+6+6+6=54; 7+7+7+7+7+7+7+7+7=63; 8+8+8+8+8+8+8+8+8=72, and 9+9+9+9+9+9+9+9+9=81. Having added the nine figures of each expression, we say, 9 times 2 is 18; 9 times 3 is 27; 9 times 4 is 36; 9 times 5 is 45; 9 times 6 is 54; 9 times 7 is 63; 9 times 8 is 72, and 9 times 9 is 81.

24. Because 4=3+1, the sum of 3 times 4 is equal to the sum of 3 times the expression 3+1; but the sum of 3 times 4 is equal to 4+4+4, and the sum of 3 times 3+1 is equal to 3+1+3+1+3+1, or 3+3+3+1+1+1, (because when we add numbers, it matters not which of them we add first;) 4+4+4 is therefore equal to 3+3+3+1+1+1; but 1+1+1=3; consequently, 4+4+4=3+3+3+3, that is to say, 3 times 4 is the same thing as 4 times 3. In a similar manner, we might show that 5 times 7 is the same as 7 times 5, and that the same

reasoning holds good with regard to any two numbers.

25. When we add numbers together, some of which consist of two or more places of figures, care must be taken to add only those figures to each other which stand in the same place of figures; because in the formation of numbers, we advance from the right hand towards the left, expressing ten units of a certain order or place by putting a single unit in the next place on the left of it (art. 15,) and therefore we easily perceive that if we added together figures standing in different places, as the units contained in the sum of these would be of different orders, we could neither continue to advance in the formation of numbers from the right hand towards the left, nor express the value of such a sum by putting it in any place whatever.

We therefore add units to units, tens to tens, hundreds to hundreds, etc. and for every ten that we find in the sum of the units of any order, we add one to the units of the next order on the left; because (Art 15,) the value of ten units of any order is expressed by a single unit of the next order on the left.

Examples.

1. $23+42+57=122$.

To perform the addition here indicated, I first add the units of each number, and then the tens of each; thus, beginning with the left hand number, I say, 3 and 2 is 5 and 7 is 12, which is the sum of all the figures standing in the place of units, and as this sum contains 2 units and 1 ten, I place the 2 units on the right of the sign $=$, so as to leave room between the 2 and the sign for the sum of the tens, and the 1 ten I add to the tens, saying, 1 and 2 is 3 and 4 is 7 and 5 is 12: now this sum consists of 2 tens and 10 tens; the 2 tens have their proper value therefore in being placed on the left of the 2 units, and as the 10 tens are 100 (Art. 12,) the value of these is expressed by

placing 1 on the left of the 2 tens, which is merely writing the sum 12 of the tens, just as it stands, on the left of the 2 units. I have therefore 122 for the sum of the three numbers 23, 42, and 57, and I read 23 *plus* 42 *plus* 57 *equal to* 122.

2. $343+267+854+5674=7138$.

When the numbers are somewhat large, for the sake of convenience we place them under each other, so that the units may be under units, the tens under tens, etc. and having drawn a line underneath, we begin with the units and add as usual. We therefore write the numbers of the present example thus:

$$\begin{array}{r} 343 \\ 267 \\ 854 \\ 5674 \\ \hline 7138 \end{array}$$

and having drawn a line underneath, we begin with the units, saying 4 and 4 is 8 and 7 is 15 and 3 is 18; then, as 18 contains 1 ten and 8 units, we write the 8 units under the column of units, and carrying the 1 ten to the column of tens, we say 1 and 7 is 8 and 5 is 13 and 6 is 19 and 4 is 23; now this sum 23 is 2 tens and 3 units, we therefore write the 3 units under the column to which they belong, and as ten units of any order are expressed (Art. 15) by a single unit of the next order on the left, we add 2 for the 2 tens to the next column, and proceding as before, we say 2 and 6 is 8 and 8 is 16 and 2 is 18 and 3 is 21; here, for the reasons already given, we write 1 and carry 2 to the next order, saying 2 and 5 is 7, this we write underneath, which completes the operation, and we have 7138 for the sum. This sum we place on the right of the sign =, and we read the whole thus: 343 *plus* 267 *plus* 854 *plus* 5674 *equal to* 7138.

In order to prove the work, we add each column from the top downwards.

3. $353+25+6564+5+78573=85520$.

In this example, as it is of no consequence with

which number we begin the addition, for greater convenience, I place the numbers to be added thus:

$$\begin{array}{r} 5 \\ 25 \\ 353 \\ 6564 \\ 78573 \\ \hline 85520 \end{array}$$

and in order to perform the operation quickly, I say (beginning always with the units,) 3 and 4 is 7 and 3 is 10 and 5 is 15 and 5 is 20; 0 and go 2 to 7 is 9 and 6 is 15 and 5 is 20 and 2 is 22; 2 and go 2 to 5 is 7 and 5 is 12 and 3 is 15; 5 and go 1 to 8 is 9 and 6 is 15; 5 and go 1 to 7 is 8.* Wherefore the sum is 85520.

REMARK.

As 1000 is formed of the addition of ten hundreds, it is (Art. 23) ten times 100; but (Art. 24) ten times 100 and 100 times 10 are the same thing; we may therefore consider 1000 either as representing 10 hundreds, 100 tens, or 1000 single units. Hence, we say that a thousand when compared to ten is a hundred, and when compared to a hundred it is ten. In the same manner, when we compare the value of a *dollar* to that of a *dime* it is ten, when we compare it to that of a *cent* it is a hundred, and when we compare it to that of a *mill* it is a thousand.

Hence, any number consisting of two or more figures which has a cipher in the place of units may be considered as representing a number of tens.

* The grammatical propriety of the expression two plus three is five, two more three is five, or two and three is five, I leave to the decision of J. Horne Tooke and other grammarians; I would only observe here, that when I make use of the expression 3 and 4 is 7, I mean that the *sum* of 3 and 4 is 7. Also, when I say 3 times 4 is 12, I mean that the *sum* or *product* is 12. This enunciation, though different from that sometimes given in books, is used in practice by the most eminent mathematicians.

Also, any number consisting of two or more figures, which has a significant figure in the place of units, may be considered as representing a number of tens and units. Thus, 540, considered as representing tens, is 54, also, 235 is 23 tens and 5 units.

4. $69+66+89+65+98+69+68+89+99+97+59+49+88=1005$.

Having placed the numbers under each other, I find that the sum of the column of units is 105, now this number is 10 tens and 5 units, I therefore write the 5 units under units, and carry the 10 tens to the column of tens: having added this column, I find that the sum is 100, which sum I write altogether on the left of the 5 units; but from the above remark, we have seen that 100 tens are equal to 1000, the figure 1 which expresses the hundred should therefore stand in the place of thousands, which it does in effect. I have therefore 1005 for the sum of the given numbers, which I place on the right of the sign $=$, as before.

5. $19+9+9+9+9+9+9+9+9+9=100$.

It will be seen in this example, that (because $9+1=10$,) in adding 9 to any significant figure of a number, that figure will be diminished and the next figure on the left increased each by a unit: thus, $19+9=28$; $28+9=37$; $37+9=46$; $46+9=55$; $55+9=64$, etc. where, in each new addition, the unit figure is one less, and the figure in the place of tens one greater.

6. $7+8+6+5+9+8+7+8+2+4+6+8+9+7+3+6+9+8+2+6=128$.

In adding numbers, whenever two figures succede each other, the sum of which is ten, it will facilitate the operation to comprehend them both together; thus, in the present example, beginning at the left, I say, 7 and 8 is 15 and 6 is 21 and 5 is 26 and 9 is 35 and 8 is 43 and 7 is 50 and 10 (comprehending $8+2$) is 60 and 10 (comprehending $4+6$) is 70 and 8 is 78 and 9 is 87 and 10 (comprehending $7+3$) is 97 and

6 is 103 and 9 is 112 and 10 (comprehending 8+2) is 122 and 6 is 128.

7. Required the sum of eight thousand and forty; ten thousand and fourteen; and four millions one thousand nine hundred and fifty-six.

Answer. Four millions twenty thousand and ten.

8. Required the sum of forty-seven billions, one million and forty; one hundred millions seven hundred and seven thousand four hundred; and thirty billions, ten millions, seventy thousand and four.

Answer. Seventy-seven billions, one hundred and eleven millions, seven hundred and seventy-seven thousand, four hundred and forty-four.

9. Required the sum of six hundred and fifty-seven; nine hundred and eighty-five; seven hundred and ninety-three; eight hundred and ninety-nine; two thousand seven hundred and ninety-five; eight thousand and forty; ninety-seven thousand and sixteen; seventy-eight thousand five hundred and nine; one hundred thousand three hundred and six; and one million seven hundred and ten thousand.

Answer. Two millions.

10. Required the sum of seven thousand two hun-hundred and eighteen; seven hundred and fifty-six; eight hundred and thirty-two; nine hundred and fifty-four; thirty-nine; five hundred and eighty-eight; seventy-seven; sixty-five; eighty-two; ninety-six; forty-seven, and sixty-four.

Answer. Ten thousand eight hundred and eighteen.

11. Required the sum of thirty-four; twenty-nine; seventy-three; eighty-five; sixty-seven; five hundred and eighty-three; eight hundred and sixty-seven; seven hundred and eighty-five; nine thousand nine hundred and eighty-six; seven thousand and forty-six and eight thousand four hundred and fifty-five.

Answer. Twenty-eight thousand and ten.

12. $37+64+86+92+48+74+96+88+97=$

13. $69+83+75+49+99+63+87+35+55+86+84=$

14. 637+24+89+56+67+82+98+38+57+42+76+93=

15. 28+394+856+58+35+94+79+87+65+99+59+67+44=

3104	2769	6385
8463	3254	4187
4757	6327	5271
5744	2128	8634
1536	6745	5862
5242	3672	4728
4255	7871	1365

85346	42139	67295
24583	73325	34823
56178	41376	52716
94236	52854	61854
82451	36272	33247
75416	26674	65176
43821	58623	47283
35763	47145	38145
17548	63727	66752

92964	86217	48912
52836	24516	71232
45384	73182	85617
52731	45923	24324
81273	62075	56163
47163	75483	28767
54615	26817	14382
47268	54076	75675
19726	37924	43836

QUESTIONS ON SECTION 3.

1. What is addition?
2. What do we call the result of an addition?

3. What sign is placed between numbers to show that they are to be added together?

4. What sign is placed between two quantities to show that they are equal to one another?

5. What is meant by the expression 5 times 7?

6. How do we prove that 3 times 4 is equal to 4 times 3, etc.?

7. In adding numbers consisting of several places of figures, why do we add units to units, tens to tens, etc?

8. Is it essential that such numbers should be placed under each other to be added, or is this done for the sake of convenience?

9. In what order do we place numbers consisting of several places of figures under each other to add them?

10. Why do we add a unit to the next order on the left for every ten which we find in the sum of any order?

11. Does every significant figure in any place on the left of units represent an exact number of tens?

12. Does every number consisting of several places either represent a number of tens, or a number of tens and units?

13. Why do we begin addition with the units?

14. How do we prove an addition?

SECTION 4.

SUBTRACTION.

26. The word *subtraction*, from two Latin words, *sub* and *traho*, signifies to take out or to take away.

The operation is that by which we take from one number as many units as are signified by a smaller number; it therefore shows what number the one contains more than the other, which number is called *remainder*, *excess*, or *difference*.

For example, if we take 3 from 9 there will be 6

left. This number 6 is therefore called the *remainder;* and as it shows by how many units 9 excedes 3, it is called the *excess,* and hence also the *difference* between the two numbers.

A line drawn thus —, which we call *minus* or *less,* is the sign of subtraction; and when placed between two numbers signifies that the *latter* is to be taken from the *former;* thus 9—3 signifies that 3 is to be taken from 9, and is read nine *minus* three; also, 9—3=6 is read nine *minus* three *equal* to six.

Examples.

1. 4—2=2; 5—2=3; 6—2=4; 7—2=5; 8—2=6, and 9—2=7.

These expressions are read, beginning at the left, 4 *minus* 2 *equal to* 2; 5 *minus* 2 *equal to* 3; 6 *minus* 2 *equal to* 4; 7 *minus* 2 *equal to* 5; 8 *minus* 2 *equal to* 6, and 9 *minus* 2 *equal to* 7. But in finding their value we say, 2 from 4 *two;* 2 from 5 *three;* 2 from 6 *four;* 2 from 7 *five;* 2 from 8 *six,* and 2 from 9 *seven.*

Here observe, that the difference between two numbers added to the less, equals the greater. For, in finding the value of the expression 4—2, we took two units from 4, and 2 were left; now if we add the 2 units taken away to the 2 units left, we shall again have 4, and so for the other expressions. Thus 2+2=4; 3+2=5; 4+2=6; 5+2=7; 6+2=8, and 7+2=9. *Hence we see that subtraction may always be proved by addition.*

2. 5—3=2; 6—4=2; 7—5=2; 8—6=2, and 9—7=2.

By comparing these expressions severally with the second, third, fourth, fifth, and sixth of those in the preceding example, we see that in taking the difference between any two numbers from the greater, we have the less. *Therefore, we can always prove a subtraction by another subtraction.*

In effect, we have seen from the proof of the first

example, that the greater number is the sum of the *difference* added to the *less;* therefore either of these taken from their sum leaves the other.

3. 7−3=4; 8−5=3; 9−5=4; 10−6=4; 10−5=5.

Here we say 3 from 7 *four;* 5 from 8 *three;* 5 from 9 *four;* 6 from 10 *four,* and 5 from 10 *five.* We must not, however, forget to read the expressions 7 *minus* 3 *equal to* 4, etc.

If we ask the question, How many added to 3 will make 7? It is evident that it will take as many units as 7 contains more than 3; but this is the difference between the numbers: we therefore subtract 3 from 7, and the answer is 4. This 4, therefore, answers to all the following questions:

What is the difference between 7 and 3?

By what number does 7 excede 3?

If 3 be taken from 7, what will be the remainder?

How many added to 3 will make 7?

And all these questions, as well as the answer, are contained in the expression 7−3.

4. 11−5=6; 12−5=7; 13−5=8, and 14−5=9.

For the sake of practice, (see Ex. 2,) say, 5 from 11 *six,* and 6 from 11 *five;* 5 from 12 *seven,* and 7 from 12 *five;* 5 from 13 *eight,* and 8 from 13 *five;* 5 from 14 *nine,* and 9 from 14 *five.*

Let the same be done with the two following examples:

5. 13−6=7; 14−6=8; 15−6=9; 11−4=7, and 13−4=9.

6. 15−7=8; 16−7=9; 17−9=8; 12−9=3, and 10−2=8.

7. 4−2=2; 6−3=3; 8−4=4; 10−5=5; 12−6=6; 14−7=7; 16−8=8, and 18−9=9.

In each of these expressions, the same number remains that we subtract, which shows that this number is contained exactly twice in the number subtracted from. We can therefore, by subtraction, find *how often* one number contains another. For ex-

ample, if we would find by subtraction how often 9 contains 3, we procede thus: 9—3=6; 6—3=3, and 3—3=0, where, having made three subtractions, we have 0 for the remainder, which shows that 9 contains 3 exactly 3 times.

27. When the numbers consist of several places of figures, we must be careful to subtract those figures of the smaller number from those of the greater which stand in the same place of figures. For, when we take one figure from another, the figure from which we subtract is diminished (art. 26,) by a number of its units equal to the number of units contained in the figure subtracted; if, therefore, the units of the figure from which we subtract be either greater or less than those of the figure subtracted, (which they must be if the figures stand in different places,) it is evident, that in performing such a subtraction, we shall take away more or less than we intend.

To avoid error, therefore, we place the less number under the greater, so that units may be under units, tens under tens, etc. as in addition, and having drawn a line underneath, we subtract each figure of the less number from the corresponding one of the greater, writing each remainder under the figure which gave it.

EXAMPLES.

1. 8984—532=8452.

To perform the subtraction here indicated, I place the less number under the greater, thus:

$$\begin{array}{r} 8984 \\ 532 \\ \hline 8452 \end{array}$$

and beginning with the units, I say, 2 from 4 leaves 2, which I write under units; proceding to the tens, I say, 3 from 8 leaves 5, which I write under tens; at the third place, I say, 5 from 9 leaves 4, which I

write under hundreds; lastly, at the fourth place, as there are no thousands to subtract, I say, 0 from 8 leaves 8, which being written underneath, completes the operation, and I have 8452 for the remainder, which I place as usual on the right of the sign.

Let the following subtractions be proved (art. 26) both by addition and subtraction.

2. 7967—3843=4124, and 5728—516=5212.

3. 9854—9823=31, and 75868—3428=72440.

28. We have seen (art. 26, Ex. 2,) that the expressions 5—3, 6—4, 7—5, 8—6, and 9—7 are each equal to 2. Now since these expressions are each equal to the same thing, they are equal to one another; therefore, 5—3=6—4, and thus we find that when we add a unit to each of the numbers 5 and 3, this addition has no effect upon their difference: again, because 5—3=7—5, if we add two units to each, their difference is still the same. Hence, *if we add the same number of units to each of two numbers, this addition will have no effect upon their difference*, because in the subtraction we take this additional number from itself, which cannot leave a remainder.

Note. The sign > signifies, when placed between two numbers, that the number to which the opening of the lines is turned is greater than the other. Thus, 9>8 signifies that 9 is greater than 8, and 8<9 signifies that 8 is less than 9.

29. As 1000>999, it is evident that some or all of the figures of the less number may be greater than the corresponding figures of the greater. Whenever, therefore, the figure which we would subtract is greater than the one above it, (and of course the subtraction of it impossible,) we add ten to the upper figure, and subtract the lower figure from the sum, writing the remainder underneath; after which, proceding towards the left, we add a unit to the next lower figure, and subtract as before.

EXAMPLES.

1. 63042—8957=54085.

I place the numbers as usual thus:

$$\begin{array}{r} 63042 \\ 8957 \\ \hline 54085 \end{array}$$

and as 7>2, I add 10 to 2, which makes 12; then, 12—7=5, which I write underneath. Proceding towards the left, I add 1 to 5, which makes 6; then, as 6>4, I add 10 to 4, which makes 14; I then say, 6 from 14 leaves 8, which I write underneath. Proceding to the third place, I add 1 to 9, which makes 10; then, as 10>0, I add 10 to 0, which gives 10, and as 10—10=0, I write 0 underneath. At the fourth place, I add 1 to 8, which makes 9, then as 9>3, I add 10 to 3, which gives 13, and as 13—9=4, I write 4 under 8. Lastly, I subtract the unit, which I should have added to the next lower figure, if there had been another, from 6, and there remains 5, which I write underneath, and have 54085, for the difference, which I place on the right of the sign.

Observe with particular attention, that in performing the above subtraction, the unit which we add to the figure 5 in the lower number, being in the place of tens, is (art. 7,) equal to the ten units previously added to the figure 2 in the upper number; the unit added to the figure 9 in the lower number, being in the place of hundreds, is (art. 12,) equal to the ten tens added to the figure 4 in the upper number; the unit added to the figure 8 in the lower number being in the place of thousands, is (art. 15,) equal to the ten hundreds previously added to 0 in the upper number; and lastly, that the unit which we take from the figure 6 in the upper number, being *ten thousand*, is (art. 15,) equal to the ten thousands which were added to the figure 3 in this same number.

Hence we see, that in subtracting from the greater number the several units which we add to the lower figures, we take from the greater number as many units as were contained in the tens, each of which was added to a figure standing one place farther towards the right, and consequently this procedure can have no effect upon the difference.

2. 63042—54085=8957.

The addition of ten to any single figure is nothing else (art. 7 and 8,) than placing a unit on the left of it; therefore, when the figure to be subtracted is greater than that above it, we read the upper figure as if it had a unit on the left of it, and having subtracted, we carry one to the next lower figure as usual. Thus, in the present example, having placed the numbers under each other,

$$\begin{array}{r} 63042 \\ 54085 \\ \hline 8957 \end{array}$$

I say, 5 from 12 *seven*; 1 to 8 is 9, and 9 from 14 *five*; 1 to 0 is 1, and 1 from 10 *nine*; 1 to 4 is 5, and 5 from 13 *eight*; and lastly, 1 to 5 is 6, and 6 from 6 *nought*, which last, however, I do not write, as it would be of no use.

3. 390605—98297=292308.

Having placed the given numbers thus,

$$\begin{array}{r} 390605 \\ 98297 \\ \hline 292308 \end{array}$$

I say, 7 from 15 *eight*; 1 to 9 is 10, and 10 from 10 *nought*; 1 to 2 is 3, and 3 from 6 *three*; here, as I did not add any to the 6, I must not add any to the next lower figure 8; I therefore continue, saying, 8 from 10 *two*; 1 to 9 is 10, and 10 from 19 *nine*; lastly, 1 to 0 is 1, and 1 from 3 *two*.

Having written successively all the numbers expressed by the words in italic, I have 292308 for the difference.

Let this, as well as all the succeding examples be proved both by addition and subtraction.

4. 400062—90807=
5. 8370060—5943007=
6. 9510587—6800769=
7. 100108058—99017349=
8. 83401263—4301829=
9. 16139724—7209815=
10. 43110832—8400853=
11. 10000000—90007=
12. 302137291—9096392=

13. From ninety-seven thousand and forty-three, take eight thousand and fifty-five.

Answer. Eighty-eight thousand nine hundred and eighty-eight.

14. From eleven millions fourteen thousand three hundred and one, take nine hundred and four thousand five hundred and three.

Answer. Ten millions one hundred and nine thousand seven hundred and ninety-eight.

15. Required the difference between the *sum* and difference of the answers to examples 7 and 8, article 25.

Answer. Eight millions forty thousand and twenty.

30. The sign + plus is called the positive sign, and the sign — minus the negative sign; accordingly, a number having the sign *plus* prefixed to it, is called a *positive number*, and a number having the sign *minus* prefixed to it, is called a *negative number*. Numbers having no sign prefixed to them are *positive*.

These two signs *plus* and *minus*, that is to say, the two operations of addition and subtraction, destroy the effect of each other. Thus, $5+3-3=5$; where having taken away the same that we added, the number 5 is still the same.

The expression $5+3-3$ is the same as $5-3+3$ or $3-3+5$, etc. for in the first, $5+3=8$, and $8-3=5$; in the second, $5-3=2$, and $2+3=5$; and in the third, $3-3=0$, and $0+5=5$. Hence it is of no

consequence in what order the numbers are placed, provided that they still retain the same sign.

When there are several positive and several negative numbers in the same expression, add all the positive numbers together, and all the negative numbers together, the difference between the two sums, preceded by the sign of the greater, will be the value of the expression.

For example, to find the value of the expression $16-9+7-5+3$, we procede thus :

$$\left.\begin{array}{l} \;\;16 \\ +\;\;7 \\ +\;\;3 \\ \hline +26 \text{ Sum of the positive numbers.} \\ \hline \\ -\;\;9 \\ -\;\;5 \\ \hline -14 \text{ Sum of the negative numbers.} \\ \hline \end{array}\right\} \begin{array}{r} +26 \\ -14 \\ \hline +12 \text{ Req. val.} \\ \hline \end{array}$$

Wherefore $16-9+7-5+3=12$.

Or, as it is of no consequence in what order the numbers are placed, we may procede thus :

$16-9+7-5+3=16+7+3-9-5=26-14=12$.

Again, to find the value of $35-24-96+18$, proceding as before, the operation stands thus :

$$\left.\begin{array}{l} +35 \\ +18 \\ \hline +53 \text{ Sum of the positive numbers.} \\ \hline \\ -\;\;24 \\ -\;\;96 \\ \hline -120 \text{ Sum of the negative numbers.} \\ \hline \end{array}\right\} \begin{array}{r} -120 \\ +\;\;53 \\ \hline -\;\;67 \text{ Re. val.} \\ \hline \end{array}$$

Wherefore $35-24-96+18=-67$. In this example, the sum of the negative numbers being the

greater of the two, the value of the expression is negative.

EXAMPLES.

1. $7+8+5-6=20-6=14$.

2. $37-48+9-7+10=37+9+10-48-7=56-55=1$.

3. $254+679-46-75+843-997=$

4. $303-27+124+500-809-99=$

5. $813+658-964-1056+549=$

6. $738-231+1090+167-46-369=$

7. $96+509-24-134+461-2678=$

8. $4031-148+27-635+27+36093-9677=$

9. $14+363-179+11-44+19+67-391+140=$

10. $11022+181-12020-15643-53+83+306=$

QUESTIONS ON SECTION 4.

1. From what is the word subtraction derived, and what do we understand by it?

2. What is the result of the operation called?

3. What sign is placed between two numbers to show that one is to be diminished by as many of its units as are contained in the other?

4. When the sign minus is placed between two numbers, which of the two is to be taken from the other?

5. When the given numbers are large, is it essential that the less should be placed under the greater, or is this done for the sake of convenience?

6. Can we perform the subtraction with nearly equal facility if we place the greater number under the less?

7. In what order do we place the numbers under each other?

8. Why do we take units from units, tens from tens, etc.?

9. If when the less number is placed under the greater, some of the lower figures are greater than those above them, what must be done?

10. How do we assure ourselves that the addition of ten units to the upper figure, and of a single unit to the next lower figure, has no effect upon the difference between the two given numbers? Explain this by an example.

11. How is subtraction proved by addition, and how by another subtraction?

12. What do we observe in comparing the signs *plus* and *minus*, or the operations of addition and subtraction with each other?

13. What is a number called when preceded by the sign plus, and what when preceded by the sign minus? To which of these does it belong when it has no sign before it?

14. How do we find the value of an expression consisting of several positive and several negative numbers?

Section 5.

MULTIPLICATION.

31. The term *multiplication* is derived from two Latin words, *multus*, many, and *plico*, I fold.

The operation is a method of finding the sum produced by the addition of a number to itself a certain number of times, more promptly than by the actual performance of this addition in the usual way. For example, to multiply 4 by 3, signifies to find the sum that would be produced by adding 4 to itself 3 times, or more exactly, by the addition of 3 fours, without performing this addition. Now it is evident that this can only be done by the recollection of the sum of 3 times 4: therefore, in order to multiply 4 by 3, we have recourse to what has been said, (art. 23,) and say 3 times 4 is 12.

The number to be multiplied is called *multiplicand*. That which shows how many times it is to be added is called *multiplier*; and the sum is called *product*. The multiplicand and multiplier are also called *factors* of the product.

32. The multiplication of the highest numbers requires nothing more than the multiplication of a single figure by a single figure.

The numbers produced by the addition of each of the nine digits to itself in the manner shown, (art. 23,) form the following table, which is attributed to Pythagoras.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The first line of this table is formed by the addition of 1 to itself.

The second by the addition of 2 in the same manner.

The third by the addition of 3, and so on for the rest.

The product of any two single figures is found by means of this table, thus: we seek one of them in the upper line, and descend from this vertically till we come opposite to the other in the first column; the number upon which we rest is the product. To find, for example, the product of 6 times 9, or the sum of 6 nines, I descend from 9 taken in the first line, till opposite 6 in the first column; the number upon which I rest is 54, which is the number sought.

EXTENSION OF THE MULTIPLICATION TABLE.

The Pythagorean Table is sufficient for the multiplication of all numbers, yet an acquaintance with the following extension of it may be found very advantageous in many cases. The student may, however, make use of it or not, as he pleases: I will only remark, that arithmeticians generally multiply by each of the numbers 11 and 12 as by a single figure.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400

33. The sign $\times$, which is called *into*, is placed between numbers to show that they are to be multiplied together: for example, 4×3 signifies that 4 and 3 are to be multiplied together, and is read 4 *into* 3; also, $4\times3=12$ is the operation performed, which is read 4 *into* 3 *equal to* 12.

Examples.

1. $2\times2=4$; $2\times3=6$; $2\times4=8$; $2\times5=10$; $2\times6=12$; $2\times7=14$; $2\times8=16$; $2\times9=18$.

We have seen (art. 24,) that 3 times 4 and 4 times 3 are the same thing; therefore, in finding the value of the expression 2×3, we may either say twice 3 is 6, or 3 times 2 is 6; and it is the same with the succeding expressions. For greater practice, we shall therefore multiply both ways, saying twice 4 is 8, and 4 times 2 is 8, etc. We may also compare the above expressions 2×2, 2×3, 2×4, etc. with their equivalent expressions $2+2$, $2+2+2$, $2+2+2+2$, etc. (art. 23. Ex. 1.)

2. $3\times3=9$; $3\times4=12$; $3\times5=15$; $3\times6=18$; $3\times7=21$; $3\times8=24$; $3\times9=27$.

We shall procede with the expressions contained in this and the succeding examples, exactly as with those of Ex. 1; we must not, however, forget to read them 3 *into* 3 *equal to* 9, etc.

3. $4\times4=16$; $4\times5=20$; $4\times6=24$; $4\times7=28$; $4\times8=32$; $4\times9=36$.

4. $5\times5=25$; $5\times6=30$; $5\times7=35$; $5\times8=40$; $5\times9=45$.

5. $6\times6=36$; $6\times7=42$; $6\times8=48$; $6\times9=54$.

6. $7\times7=49$; $7\times8=56$; $7\times9=63$.

7. $8\times8=64$; $8\times9=72$, and $9\times9=81$.

34. When the multiplicand consists of several figures, and the multiplier of a single figure, we place the multiplier under the unit figure of the multiplicand; and having drawn a line underneath, we multiply successively all the figures of the multiplicand by the multiplier, beginning with the units.

The unit figure of each product we write under the figure which gave it, and the tens we retain in the same manner as in addition, to add them to the next product. Having multiplied the last or left hand figure of the multiplicand, we write the whole product underneath, which completes the operation, the number under the line being the product of the multiplicand and multiplier, that is to say, the sum which would arise from the addition of as many times the multiplicand as there is a unit in the multiplier.

Examples.

1. To multiply 6342 by 6, I write the numbers thus:

$$\begin{array}{r} 6342 \\ 6 \\ \hline 38052 \end{array}$$

and beginning with the units, I say, 6 times 2 is 12. Here, as in addition, I write 2 and carry 1: I then say, 6 times 4 is 24, and 1 is 25; I write 5, and carry 2: then, 6 times 3 is 18, and 2 is 20; I write 0, and carry 2: lastly, 6 times 6 is 36, and 2 is 38, the whole of which I write underneath, and I have 38052 for the product of 6342 multiplied by 6.

If we write the number 6342 six times, thus,

$$\begin{array}{r} 6342 \\ 6342 \\ 6342 \\ 6342 \\ 6342 \\ 6342 \\ \hline 38052 \end{array}$$

and perform the addition, we have the same result.

2. $86542 \times 5 = 432710$.

Having placed the numbers thus,

$$\begin{array}{r} 86542 \\ 5 \\ \hline 432710 \end{array}$$

in order to perform the operation quickly, I say, 5 times 2 is 10; *nought* and go 1: 5 times 4 is 20, and 1 is 21; 1 and go 2; 5 times 5 is 25, and 2 is 27; 7 and go 2: 5 times 6 is 30, and 2 is 32; 2 and go 3; 5 times 8 is 40, and 3 is 43.

3. $962 \times 3 = 2886$; $875 \times 3 = 2625$, and $3469 \times 3 = 10407$.

4. $732 \times 4 = 2928$, and $96549 \times 4 = 386196$.

5. $6097 \times 5 = 30485$, and $54038 \times 5 = 270190$.

6. $3927 \times 6 = 23562$, and $85976 \times 7 = 601832$.

7. $896345 \times 8 = 7170760$, and $985647 \times 9 = 8870823$.

8. Multiply 5384679 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, and find the sum of the several products.

Answer. Two hundred and thirty-six millions, nine hundred and twenty-five thousand, eight hundred and seventy-six.

9. Multiply 93578864 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, and find the difference between the sum of the products and one trillion.

Answer. Nine hundred and ninety-five billions, eight hundred and eighty-two millions, five hundred and twenty-nine thousand nine hundred and eighty-four.

35. When the multiplier, as well as the multiplicand, consists of several figures, having placed the numbers under each other, and drawn a line underneath the whole, first multiply all the figures of the multiplicand by the unit figure of the multiplier, as above. Then multiply by the tens, placing the product under the first product, so that its unit figure may stand under the tens of the first product. Continue to multiply successively by all the figures of the multiplier, always placing the first figure of each product under that figure of the first product which stands in the same place of figures as the figure by which you multiply. Having multiplied by all the figures of the multiplier, add the several products

which they have given, and the sum is the total product.

For example, to find the product of 6574, multiplied by 432, I place the numbers thus,

6574	
432	
13148	twice the multiplicand.
19722	10 times 3 times, or 30 times —.
26296	100 times 4 times, or 400 times —.
2839968	Product, or 432 times 6574.

and having multiplied by the unit figure 2 as usual, I multiply by the 3 tens in the same manner, writing the first figure of this second product in the place of tens, that is to say, under the tens of the first product; lastly, I multiply by the 4 hundreds, writing the first figure of this product under the place of hundreds in the first product.

Now the second product is 3 times the multiplicand, but because the units of which any figure is composed are (art. 15,) ten times as great as they would be if the figure stood one place farther towards the right, the second product, in the place where it now stands, is just ten times as great as if it had stood directly under the first; it is therefore ten times 3 times, or 30 times the multiplicand. For the same reason, because the third product, which is 4 times the multiplicand, is removed two places towards the left, it is ten times ten times, or 100 times as great as if it had stood directly under the first; it is therefore 400 times the multiplicand. Wherefore, because $400+30+2=432$, the sum of these three products, added just as they stand, is 432 times the multiplicand; it is therefore the product of 6574 multiplied by 432, as was required.

When several numbers have the sign $\times$ *into* between them, first multiply any two of them; multiply the product by the next number, and this pro-

duct again by the next, and so on till all the numbers are involved.

Ex. $7\times4\times2\times6=336$, which is found, beginning at the left hand, thus: $7\times4=28$; then $28\times2=56$, and lastly, $56\times6=336$; or, beginning at the right hand, thus: $6\times2=12$; $12\times4=48$, and $48\times7=336$. It is therefore of no consequence as to the result, in what order the numbers are multiplied. In the same manner, let the numbers in the following examples be multiplied from left to right, and from right to left; if the product is the same both ways, the work can scarcely be wrong.

EXAMPLES.

1. $46\times34\times87=136068$.
2. $26\times53\times47\times9=$
3. $57\times73\times64\times4=$
4. $384\times58\times97=$
5. $876\times239\times547=$
6. $4378\times534\times6892=$
7. $5276\times8439\times79584=$

36. When there are ciphers between the figures of the multiplier, we omit them, and in every other respect procede as before. For example, if we would multiply 49765 by 7008, I place the numbers thus:

```
      49765
       7008
     ------
     398120  8 times the multiplicand.
   348355    7000 times  ———.
  ---------
  348753120  Sum, or 7008 times.
```

and having multiplied by the unit figure 8 as usual, omitting the ciphers, I directly procede to the 7, and as this figure stands in the place of thousands, in multiplying by it, I place the first figure of this last product under the thousands of the first, by which means the last product becomes 7000 times the mul-

tiplicand; wherefore, having added the two products together, I have 348753120 for the total product. Let the numbers in the following examples be multiplied from left to right, and from right to left.

Examples.

1. $603 \times 307 \times 2004 = 370982484$.
2. $35007 \times 9008 \times 706 =$
3. $40084 \times 506 \times 50009 =$
4. $80003 \times 7006 \times 8904 =$
5. $57004 \times 60009 \times 12007 =$

37. If either or both of the factors terminate with ciphers, we omit the terminating ciphers in multiplying, and place them all on the right of the product. For example, to multiply 3700 by 150, I place the numbers thus:

```
  3700
   150
 ------
  185
  37
 ------
555000
```

after which I multiply 37 only by 15, and have 555 for the product. But 37 is a number of hundreds, and 15 a number of tens; the product 555 is therefore tens of hundreds, that is to say, thousands; I therefore place three ciphers on the right of it, which are as many as are found on the right of both the factors. The same reasoning may be applied to every case of this kind. Let the following multiplications be performed both ways, as in the preceding articles.

Examples.

1. $320 \times 600 \times 780 = 149760000$.
2. $50800 \times 25000 \times 700 =$
3. $4902000 \times 8050 \times 3400 =$

4. $2879000 \times 300090 \times 40500 =$

5. $9200 \times 802006000 \times 50300 =$

38. The units composing any figure towards the left are (art. 15,) ten times as great as they would be if the figure stood one place farther towards the right: therefore, to multiply a number by 10, 100, 1000, etc. we have only to place one, two, three, etc. ciphers on the right of it. Thus, $679 \times 10 = 6790$; $679 \times 100 = 67900$; $679 \times 1000 = 679000$, etc.

Remark.

From the Pythagorean Table, it is evident, that in most cases, the product of two numbers contains as many figures as are contained in both factors; and from the above example, because 10, 100, 1000, etc. are the least numbers that can contain 2, 3, 4, etc. figures, we easily perceive that the number of figures contained in both factors can never excede the number contained in the product by more than one figure. Lastly, as $10 > 9$; $100 > 99$; $1000 > 999$, etc., it is evident that the product can never contain more figures than are contained in both factors. Therefore, *the product of two numbers always contains as many figures as are contained in both, or at most but one less.*

CONVENIENT CONTRACTIONS.

39. To multiply any number by any of the numbers 11, 12, 13, etc. to 19, inclusive.

If we would multiply 5734 by 11, I write the multiplicand with a line underneath, thus:

$$\begin{array}{r} 5734 \\ \hline 63074 \end{array}$$

and commencing with the unit figure 4, I write it under the line: then proceding regularly towards the left, I add each figure to the next figure on the right of it; thus: 3 and 4 is 7, which I write under 3: then

7 and 3 is 10; I write 0 under 7, and carry 1 to 5 which makes 6: then 6 and 7 is 13; I write 3 under 5, and carry 1, which, as there are no more figures, I add to the 5, and placing the sum 6 in the product one place farther to the left, I have 63074 for the product of 5734 multiplied by 11. The reason of this will easily be understood, as follows:

57340	is ten times the multiplicand.
5734	is once ———.
63074	Sum, or 11 times the multiplicand.

Observe, that in placing any number under ten times that number, its unit figure stands under a cipher, and that each of the other figures stands under the next figure on the right of it; also, that the last figure is projected one place farther towards the left.

To multiply 2568 by 14, I place the numbers thus:

$$\begin{array}{r} 2568 \\ 14 \\ \hline 35952 \end{array}$$

and beginning with the units, I say, 4 times 8 is 32; I write 2 and carry 3: then 4 times 6 is 24, and 3 is 27; to this I also add the figure 8 on the right, which makes 35; I write 5 and carry 3: I then say, 4 times 5 is 20, and 3 is 23, and 6 (always adding the right hand figure) is 29; I write 9 and carry 2: then 4 times 2 is 8, and 2 is 10, and 5 is 15; I write 5 and carry 1, which, as there are no more figures, I add to the 2, and placing the sum 3 on the left, I have 35952 for the required product.

The reason of this operation is the same as that given for the multiplication by 11. For, as we there added once to ten times, we have here in like manner added four times to ten times. We should proceed in the same manner with any of the multipliers above mentioned.

TO MULTIPLY A NUMBER BY NINE, OR ANY NUMBER OF NINES.

It is evident, that if we place once any number under ten times that number, and subtract, we shall have nine times that number.

Wherefore, to multiply 3786 by 9, I write the number with a line underneath, thus:

```
 3786
-----
34074
```

and beginning with the unit figure, I say, 6 from 10 leaves 4; I write 4 under 6, and carry one to the next figure on the left: then 1 and 8 is 9, and as I cannot take 9 from 6, I say 9 from 16 leaves 7, which I write under 8, and carrying 1 to 7 which makes 8, I say 8 from 8 leaves 0, which I write under 7: then 3 from 7 leaves 4, which I write under 3: lastly, I place 3 in the product on the left, and I have 34074 for the required product. This operation is the same as the following:

```
37860 is ten times the multiplicand.
 3786 is once ———.
-----
34074 difference, or 9 times the multiplicand.
```

To multiply by a number of nines. Place as many ciphers on the right of the given multiplicand as there are nines to multiply by; from this new number, subtract the multiplicand, and the remainder will be the required product.

For example, to multiply 6932 by 999,

```
I write,    6932000
               6932
            -------
            6925068
```

In placing three ciphers on the right, I multiply by 1000 (art. 38;) then as $1000-1=999$, in subtracting once the number from a thousand times, the remainder is 999 times, as was required.

A number that is a certain number of times another number, is called a multiple of that other number. Thus 12 is a multiple of 3, because it contains 3 a certain number of times exactly; also, a number that is a multiple of several numbers, is called a common multiple of those numbers; thus 12 is a common multiple of 6, 4, 3, and 2.

40. In the multiplication of two numbers, when any part of the multiplier towards the left is a multiple of the part preceding it, the work may often be considerably shortened, as will be seen in the following examples.

Example 1.

If we have 34625 to multiply by 248, I write the numbers as usual,

```
  34625
    248
-------
 277000  eight times the multiplicand.
831000   two hundred and forty times.
-------
8587000
```

and having multiplied by the unit figure 8 of the multiplier, I have 277000 for the first partial product. Then as the part 24 of the multiplier is a multiple of 8, since it contains 8 just 3 times, instead of multiplying by the figures 24 as usual, I multiply the first product 277000, which is 8 times the multiplicand, by 3, and I have 8587000 for 24 times the multiplicand; the unit figure of the last product being placed under the tens of the first, makes the whole of this last product signify 240 times the multiplicand: wherefore, the sum of the two products is 248 times the multiplicand, as was required.

Example 2.

What is the product of 358976 multiplied by 147497?

```
    358976
    147497
   -------
   2512832   seven times the multiplicand.
  17589824   seven times seven times, or 49 times.
52769472     three times 49 times, or 147 times.
-----------
52947888072 Product.
```

Having multiplied by the unit figure 7 of the multiplier, I have 2512832 for the product; then, as 49 on the left is just 7 times 7, I take 7 times 2512832, which gives 17589824 for 49 times the multiplicand; this product being removed one place towards the left, is 490 times the multiplicand: again, because 147, which is on the left of 49, is just 3 times 49, I multiply the product 17589824 by 3, and I have 52769472 for 147 times the multiplicand; the unit figure of this product being put in the place of thousands, makes the whole signify 147000 times the multiplicand. Now, because $147000+490+7=147497$, the sum of these three products is 147497 times the multiplicand, as was required.

Let the value of the following expressions be found according to the above methods of contraction; also, let them be multiplied from left to right, and from right to left; if the product is found the same both ways, this will be a sufficient proof of the correctness of the operation.

3. $96817\times3216\times119=$
4. $192486\times4214\times168=$
5. $4816\times999\times93618=$
6. $7212\times1918\times3913=$
7. $84427\times1415\times12999=$
8. $7813\times13211\times162546=$
9. $15317\times19818\times16814=$
10. $243819\times999\times11216=$
11. $124\times153\times9999\times3417=$
12. $185496\times271863\times11214=$

41. A *vinculum* is a bar ———, or parenthesis ( ), used to collect several quantities into one. Thus

$\overline{6+4}\times 3$, or $(6+4)3$, signifies that the sum of 6 and 4 is to be multiplied by 3, and $\overline{6-4}\times 3$, or $(6-4)3$ signifies that the difference between 6 and 4 is to be multiplied by 3. Again, $8-\overline{4+2}$, or $8-(4+2)$ signifies that the sum of 4 and 2 is to be taken from 8; also, $8-\overline{4-2}$, or $8-(4-2)$ signifies that the difference between 4 and 2 is to be taken from 8.

Any two numbers having the sign $\times$ *into* between them, are considered as having immediate connection with each other, unless this is otherwise determined by the vinculum. Thus, $6+4\times 3=18$, and $4+6\times 3=22$; but $\overline{6+4}\times 3=30$, and $(4+6)\times 3=30$.

Also, $6-2\times 3=0$, and $(6-2)3=12$.

When an expression under a vinculum is preceded by the sign *minus*, the vinculum may be taken away, and the expression still retain the same value, by changing the signs of the numbers under the vinculum from *plus* to *minus*, and from *minus* to *plus*, the sign preceding the whole expression remaining the same. Thus,

$$36-\overline{7+3}=26\text{; also, }36-7-3=26.$$

Again, $36-(7-3)=32$; also $36-7+3=32$.

In the expression $36-\overline{7+3}$, because the numbers 7 and 3 are united by the vinculum, we understand that their sum is to be taken from 36; but if the vinculum were taken away, we should then consider the 3 as a number to be added, seeing that it is preceded by the sign plus; therefore, that we may still consider it as a number to be subtracted, in removing the vinculum, we change the sign preceding it from *plus* to *minus*.

Also, in the expression $36-\overline{7-3}$, as 7 and 3 are united by the vinculum, we understand that their difference is to be taken from 36; but if the vinculum were taken away, we should then consider both 7 and 3 as numbers to be subtracted; wherefore, in order to take away no more than their difference, having taken away 7 which is 3 too much, we must

add 3 to make up the deficiency; that is to say, in removing the vinculum, we must change the sign preceding 3 from *minus* to *plus*. It is the same, whatever is the number of quantities under the vinculum..

To multiply a number is (art. 31,) to add it to itself a certain number of times. Now, as $9=6+3$, it is evident that $9+9=6+3+6+3$, or $6+6+3+3$; that is to say, that twice 9 is equal to twice 6 plus twice 3. Again, as $9=5+4$, it is evident that $9+9+9=5+4+5+4+5+4$, or $5+5+5+4+4+4$; that is to say, three times 9 is equal to three times 5 plus three times 4. Hence we easily perceive that *when a number is multiplied by any other number, the product is equal to the sum of the products, when all its parts are separately multiplied by that number.*

For example, $8\times6=48$; and if we divide 8 into the parts 4, 3, and 1, and multiply each of these by 6, thus,

$$\begin{array}{c} 4+3+1 \\ 6 \\ \hline 24+18+6 \end{array}$$

the sum of the three products 24, 18, and 6, is 48.

Also, dividing the multiplier 6 into the two parts 4 and 2, and multiplying by each of these separately, as above,

$$\begin{array}{r} 4+3+1 \\ 4+2 \\ \hline 16+12+4 \\ 8+6+2 \\ \hline 16+12+4+8+6+2 \end{array}$$

we have $16+12+4$ for the product of four times 8, and $8+6+2$ for the product of twice 8. Now the sum of these two products should be 6 times 8, or 48, which it is in effect.

From the above, it is evident that $(8+7+5)9=72+63+45$.

Again, $(6-4)3=6$. Here we may observe that as it is the difference between 6 and 4 which is to be multiplied by 3, if we multiplied the whole 6, which is 4 too great by 3, the product will be 3 times 4 too much, and consequently will require to be diminished by 3 times 4, in order to give the true product. Now this is the same as to multiply the numbers 6 and 4 separately by 3, and to take the difference between the products. Therefore, $(6-4)3=18-12$. Also, $(5+8-2-3)4=20+32-8-12$. When the vinculum is thus removed by writing the several products, the expression is said to be expanded.

Thus the expression $76-(7-8+6-3)5$ when expanded, is $76-35+40-30+15$.

EXAMPLES SHOWING THE USE OF THE VINCULUM.

1. $32-\overline{17+6-14-9}=32$.
2. $186+43-27-\overline{6+8+4}=184$.
3. $9645-(1736+23)-17\times6=7784$.
4. $9645-1736+\overline{23-17}\times6=7945$.
5. $9645-\overline{1736+23-17}\times6=-807$.
6. $\overline{9645-1736+23-17}\times6=47490$.
7. $50-(7-3+9+8)-\overline{6+2-4}\times3=17$.
8. $50-7+3-9-8-(6+2-4)3=17$.
9. $672-\overline{81-31-136}\times8-(7+4)9=1261$.
10. $\overline{672-(81-31-136)8-(7+4)}\times9=12141$.

QUESTIONS ON SECTION 5.

1. From what is the word multiplication derived?
2. What is understood by the multiplication of a number?
3. What is the number called which we multiply?
4. What is the number called which shows how many times the multiplicand is to be repeated?

5. What is the result of the operation called?

6. What general name do we give to the multiplicand and multiplier?

7. In what does multiplication differ in its object from addition?

8. What is the sign of multiplication, and how is it formed?

9. How does this sign differ from the sign plus?

10. When the multiplicand consists of several figures, and the multiplier of a single figure, how do we procede?

11. Why do we carry a unit for every ten which we find in the product of any figure, to the product of the next figure on the left?

12. Why do we begin multiplication with the units?

13. When the multiplier also consists of several figures, why do we place the unit figure of each partial product under that figure of the first product which stands in the same place of figures as the figure by which we multiply? Let this be explained by an example.

14. When there are ciphers between the figures of the multiplier, how do we procede?

15. If either or both of the factors terminate with ciphers, how do we procede?

16. How do we multiply a number by 10, 100, 1000, etc.?

17. What do we observe with regard to the number of figures in the product?

18. What is the most expeditious method of multiplying a number by 11, 12, 13, etc. as far as 19 inclusive?

19. How do we multiply by 9, or a number of nines?

20. What is the meaning of the term multiple?

21. What is a common multiple?

22. When one part of the multiplier on the left is a multiple of that which precedes it, how do we contract the operation? Give an example.

23. What is a vinculum, and how used?

24. What is understood by the expansion of an expression comprised under a vinculum?

25. How do we expand such an expression when preceded by the sign minus? Give an example.

SECTION 6.

DIVISION.

42. We have seen (art. 26,) that we can by several subtractions find how often one number is contained in another, but by division, we find the same with much greater facility. For example, to divide 20 by 5, that is to say, to find how often 5 is contained in 20, from the recollection (art. 23,) of the number of fives which must be added together to make 20, we say 5 is contained in 20 *four* times. Therefore, 4 is the answer.

The number to be divided is called *dividend*, the number to divide by *divisor*, and the number found by the operation *quotient*. Hence, in the above example, 20 is the dividend, 5 the divisor, and 4 the quotient.

43. The sign which denotes division is formed thus $\div$. This sign is called *by*, and signifies that the former of the two numbers between which it is placed is to be divided by the latter. Thus, $24\div6$ signifies that 24 is to be divided by 6, and is read 24 *by* 6. Also, if the numbers are placed thus $\frac{24}{6}$, the same thing is signified, the number above the line being always the dividend, and that below the line the divisor. Hence $24\div6=4$, or $\frac{24}{6}=4$, each of which expressions we read 24 *by* 6 *equal to* 4. Also, $24\div4=6$. From this example, we see that $24\div6=4$, and that $24\div4=6$; but $4\times6=24$, whence we conclude that *if we divide the product of two numbers by either of those numbers, the quotient will be the other number;* therefore, if we multiply the quotient by the divisor, we shall always reproduce the divi-

dend, and consequently, *division may always be proved by multiplication.*

Also, because we find the divisor in dividing by the quotient, *one division may always be proved by another*, at least when the dividend is a multiple of the divisor.

EXAMPLES.

1. $4 \div 2 = 2$; $6 \div 2 = 3$; $8 \div 2 = 4$; $10 \div 2 = 5$; $12 \div 2 = 6$; $\frac{14}{2} = 7$; $\frac{16}{2} = 8$, and $\frac{18}{2} = 9$. We read these expressions 4 by 2 equal to 2, 6 by 2 equal to 3, etc.; but in finding their value, we say, 2 is contained in 4 twice, writing 2 for the quotient on the right of the sign; again, 2 is contained in 6 three times, writing 3 as before. As the divisor is found in dividing by the quotient, we shall for greater practice say, 2 in 8 four times, and 4 in 8 twice; 2 in 10 five times, and 5 in 10 twice, and so on for the other expressions in this and the succeding examples.

2. $\frac{6}{3} = 2$; $\frac{9}{3} = 3$; $\frac{12}{3} = 4$; $\frac{15}{3} = 5$; $\frac{18}{3} = 6$; $\frac{21}{3} = 7$; $\frac{24}{3} = 8$, and $\frac{27}{3} = 9$.

3. $\frac{8}{4} = 2$; $\frac{12}{4} = 3$; $\frac{16}{4} = 4$; $\frac{20}{4} = 5$; $\frac{24}{4} = 6$; $\frac{28}{4} = 7$; $\frac{32}{4} = 8$, and $\frac{36}{4} = 9$.

4. $\frac{10}{5} = 2$; $\frac{15}{5} = 3$; $\frac{20}{5} = 4$; $\frac{25}{5} = 5$; $\frac{30}{5} = 6$; $\frac{35}{5} = 7$; $\frac{40}{5} = 8$, and $\frac{45}{5} = 9$.

5. $\frac{12}{6} = 2$; $\frac{18}{6} = 3$; $\frac{24}{6} = 4$; $\frac{30}{6} = 5$; $\frac{36}{6} = 6$; $\frac{42}{6} = 7$; $\frac{48}{6} = 8$, and $\frac{54}{6} = 9$.

6. $\frac{14}{7} = 2$; $\frac{21}{7} = 3$; $\frac{28}{7} = 4$; $\frac{35}{7} = 5$; $\frac{42}{7} = 6$; $\frac{49}{7} = 7$; $\frac{56}{7} = 8$, and $\frac{63}{7} = 9$.

7. $\frac{16}{8} = 2$; $\frac{24}{8} = 3$; $\frac{32}{8} = 4$; $\frac{40}{8} = 5$; $\frac{48}{8} = 6$; $\frac{56}{8} = 7$; $\frac{64}{8} = 8$ and $\frac{72}{8} = 9$.

8. $\frac{18}{9} = 2$; $\frac{27}{9} = 3$; $\frac{36}{9} = 4$; $\frac{45}{9} = 5$; $\frac{54}{9} = 6$; $\frac{63}{9} = 7$; $\frac{72}{9} = 8$, and $\frac{81}{9} = 9$.

9. $\frac{2}{2} = 1$; $\frac{3}{3} = 1$; $\frac{4}{4} = 1$; $\frac{5}{5} = 1$; $\frac{6}{6} = 1$; $\frac{7}{7} = 1$; $\frac{8}{8} = 1$, and $\frac{9}{9} = 1$. Also, $\frac{2}{1} = 2$; $\frac{3}{1} = 3$, etc.

From this last example we see that any number divided by itself is equal to a unit, and that any number divided by a unit is still the same.

10. $\frac{4}{2} = 2$; $\frac{8}{4} = 2$; and $\frac{16}{8} = 2$. Because each of

these expressions is equal to 2, they are equal to one another; therefore, as $\frac{4}{2}=\frac{8}{4}=\frac{16}{8}$, etc. if we multiply both the dividend and divisor by the same number, this has no effect upon the quotient; and consequently if we divide both by the same number, the quotient will still be the same. We will exemplify this a little farther, $\frac{6}{2}=3$, and if we multiply the dividend and divisor both by 4, we have $\frac{24}{8}$, and $\frac{24}{8}=3$, as before. Again, $\frac{32}{8}=4$, and if we divide the dividend and divisor both by 4, we have $\frac{8}{2}$, and $\frac{8}{2}=4$, as before. Wherefore this is general, whatever be the dividend and divisor.

As the number 24 is the same multiple of 6, that 8 is of 2, the numbers 24 and 8 are called *equimultiples* of the numbers 6 and 2, and we have seen that these equimultiples contain each other as often as the numbers contain each other, and that this is general whatever be the numbers and the equimultiples.

44. The signs $\times$ *into* and $\div$ *by*, that is to say, the operations of multiplication and division, destroy the effect of each other. Thus $6\times4\div4=6$, where we see that having multiplied and divided by the same thing the number 6 is still the same. Division is therefore the reverse of multiplication, the latter being an easier method of performing what might be done by addition, and the former an easier method of performing what might be done by subtraction. We will now procede to the division of a number consisting of many figures by a number expressed by a single figure.

45. As 12 contains 4 three times, 12+12 will contain 4 three times plus three times; that is to say, twice 12 will contain 4 just twice as often as 12 contains 4; ten times 12 will contain 4 just ten times as often as 12 contains 4, &c. Hence it is easy to perceive that if we increase the dividend any number of times, leaving the divisor the same, the quotient will be increased the same number of times. Therefore if we divide 12 tens by four, the quotient

will be three tens; if we divide 12 hundreds by 4, the quotient will be 3 hundreds, etc.

Now, in dividing a number consisting of many figures, as it is evident from what has been said that no quotient figure of a superior order can ever arise from the division of any of the inferior orders, we always begin the division at the left hand.

Having taken a part of the dividend sufficient to contain the divisor, and placed the number of times that the divisor is contained in this part in the quotient, it is evident that for every remaining figure in the dividend we must have a figure in the quotient. For, if the part of the dividend first taken is thousands, the quotient figure found in dividing it will also be thousands, and will require as many figures on the right of it as will make it stand in the place of thousands, that is to say, just as many as remain in the dividend; for this reason, when any of the partial dividends does not contain the divisor, we must always place a cipher in the quotient.

We have seen (art. 44) that when a number is multiplied and divided by the same number it still remains the same. Therefore, having multiplied 563 by 5 and found the product 2815, it is evident that if we divide this product by 5 we shall have 563 for the quotient. In order to perform the division, we place the numbers thus:

```
          Dividend
Divisor 5)2815
          ----
           563 Quotient
```

and, as 5 is not contained in 2, the first figure of the dividend on the left, we take the two first figures, and seek how often 5 is contained in 28; we find 5 times and 3 over, but this number 28 is 28 hundreds, and consequently the quotient figure 5 that we have found is 5 hundreds, we therefore write it under 8, and as there is a remainder of 3 hundreds to be divided, we carry this to the next place of figures on the right, where (art. 15) it becomes 30, and toge-

ther with the 1 in this place is 31 tens; we then seek how often 5 is contained in 31, we find 6 times and 1 over; now, as 31 is a number of tens, the quotient figure 6 which we find in dividing it is 6 tens, we therefore write 6 under tens and carry the ten which remains to be divided to the place of units, here together with the 5 units it makes 15. Lastly, we divide 15 by 5, saying 5 in 15 three times; having written the quotient figure 3 under units the operation is finished, and we have 563 for the quotient, as was proposed.

In multiplying 563 by 5, having placed the numbers as usual, thus:

$$\begin{array}{r} 563 \\ 5 \\ \hline 2815 \end{array}$$

we say 5 times 3 is 15; we write 5 and carry one. Now this is 1 ten, and is the same ten which is brought back again to the place of units in dividing the product 2815 by 5. Again, we say 5 times 6 is 30 and 1 is 31; we write 1 and carry 3. Now this is 30 tens or 3 hundreds, and is the same that remains in dividing 28 hundreds by 5. Lastly, we say 5 times 5 is 25 and 3 is 28, which we write underneath, and we have 2815 for the product. Having reproduced the dividend, this last operation is a proof of the correctness of the first.

Also remark that the dividend is always the product of two numbers, one of which is the divisor and the other the quotient.

Ex. 2. $560799 \div 2 = 280399\frac{1}{2}$

To perform the operation indicated, place the numbers thus:

$$\begin{array}{r} 2)560799 \\ \hline 280399\frac{1}{2} \end{array}$$

and say 2 in 5, twice and 1 over; write 2 and carry 1 to the next place on the right, where with the 6 it makes 16. Then say 2 in 16, 8 times, and write 8

underneath; also say 2 in 0, 0 times, writing 0 underneath: then 2 in 7, 3 times and 1 over; write 3 and carry 1 to 9, which (art. 15) makes 19. Then say 2 in 19, 9 times and 1 over; write 9 and carry the 1, which is 10, to the 9 units. Again say 2 in 19, 9 times and 1 over; write 9, and as this last remainder 1 is a part of the dividend which has not yet been divided, place it as a dividend over the divisor 2 thus $\frac{1}{2}$, and considering it as representing the quotient of 1 divided by 2, that is to say, half a unit, place it in the quotient on the right hand as above. Do the same in all cases where there is a final remainder.

To avoid embarrassment in proving examples like the present by multiplication, we need only recollect that a number multiplied and divided by the same number is not altered (art. 44.) Thus $1 \div 2 \times 2 = 1$, or, which is the same thing, $\frac{1}{2} \times 2 = 1$. Also, since $\frac{5}{7}$ signifies the quotient of 5 divided by 7, and since the quotient multiplied by the divisor will (art. 44) reproduce the dividend, the quotient $\frac{5}{7}$ multiplied by the divisor 7 will reproduce the dividend 5, that is to say, $\frac{5}{7} \times 7$, or $5 \div 7 \times 7 = 5$; and it is the same with all similar expressions. Such expressions as these, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{5}{7}$, $\frac{8}{9}$, etc. are read *one-half*, *one-third*, *one-fourth*, *five-sevenths*, *eight-ninths*, &c. Their nature will be more fully explained hereafter.

To prove the above example, place the divisor 2 under the quotient, thus:

$$\begin{array}{r} 280399\frac{1}{2} \\ 2 \\ \hline 560799 \end{array}$$

and multiply, saying $\frac{1}{2} \times 2 = 1$ (art. 44;) and as this 1 is a unit, retain it; then say twice 9 is 18 and the unit retained is 19, 9 and go 1; again, twice 9 is 18 and 1 is 19, 9 and go 1; twice 3 is 6 and 1 is 7; twice 0 is 0; twice 8 is 16, 6 and go 1; twice 2 is 4 and 1 is 5.

Let the same method of operation and proof be

pursued with regard to each of the following examples.

3. Divide 8709120 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, and find the sum of the quotients.

Answer. Fifteen millions nine hundred and twenty-eight thousand seven hundred and four.

4. $76947 \div 2 = 38473\frac{1}{2}$ and $\frac{48637}{3} = 16212\frac{1}{3}$.

5. $106204 \div 3 = 35401\frac{1}{3}$ and $\frac{57995}{4} = 14498\frac{3}{4}$.

6. $112008 \div 4 = 28002$ and $\frac{63971}{5} = 12794\frac{1}{5}$.

7. $731344 \div 5 = 146268\frac{4}{5}$ and $\frac{11693}{6} = 1948\frac{5}{6}$.

8. $79248657 \div 7 = 11321236\frac{5}{7}$.

9. $93010681 \div 8 = 11626335\frac{1}{8}$.

10. $1065067855 \div 9 = 118340872\frac{7}{9}$.

46. When the divisor consists of several figures, we cannot procede as in the above article, that is to say, we cannot easily perform the multiplication and subtraction mentally; we shall therefore show by examples the method to be pursued in this case.

To this effect we shall first multiply 213 by 52, as follows:

```
  213
   52
 ----
  426
 1065
 -----
 11076
```

and having found the product 11076, we shall divide this product by 213, which (art. 43,) will give 52 for the quotient.

To perform this division, we place the numbers thus:

```
                  Dividend
        Divisor 213)11076(52 Quotient
                    1065
                    ----
                     426
                     426
                     ---
                     ***
```

and first we take as many of the left hand figures of the dividend as will contain the divisor, that is to say, we take four figures, because three are not sufficient. We then seek how often the divisor 213 is contained in 1107; and to find this with facility, we seek how often the hundreds of the divisor are contained in the hundreds of the part of the dividend which we have taken; that is to say, we seek how often the greater part of the divisor is contained in the greater part of this dividend: we therefore say, 2 in 11 five times; this 5 we place in the quotient on the right of the dividend.

We then multiply 213 by 5, placing the product 1065 under 1107, and having drawn a line underneath, we subtract and have 42 for the remainder.

To the right of this remainder we bring down the last figure 6 of the dividend, and seek how often 213 is contained in 426; by seeking how often 2 is contained in 4, we find twice; wherefore, we write 2 in the quotient on the right of 5.

Again, we multiply the divisor 213 by 2, placing the product 426 under 426, the part divided. Having drawn a line underneath, we subtract, and as nothing remains, we find that 11076 contains 213 just 52 times, as was anticipated.

We shall now divide 11076 by 52, which (art. 43.) will give 213 for the quotient.

```
52)11076(213
   104
   ---
    67
    52
   ---
    156
    156
    ---
    ***
```

Here, as the two first figures of the dividend are not sufficient to contain the divisor, we take the three first, and seek how often 52 is contained in 110, by

seeking how often 5 is contained in 11. It is contained twice, or 2 times: we therefore place 2 in the quotient.

We next multiply the divisor 52 by the quotient 2, placing the product 104 under 110.

Having drawn a line underneath, and subtracted, we have 6 for the remainder, to the side of which we bring down 7, the next figure of the dividend.

We then seek how often 52 is contained in 67, saying, 5 in 6 once; we therefore write 1 in the quotient on the right of 2.

Having multiplied the divisor by this 1, we subtract the product 52 from 67, which leaves a remainder of 15. To the right of this remainder we bring down the last figure 6 of the dividend.

Again we seek how often 52 is contained in 156, saying, 5 in 15 three times; we place 3 in the quotient, and having multiplied the divisor 52 by this 3, we have 156, which being subtracted from 156, the part divided leaves nothing.

We have therefore 213 for the quotient, as was anticipated. Thus we see that each of these divisions is a proof of the other, and that the multiplication with which we commenced is a proof of both.

Observe also, that the steps which we take in this operation are the same as those taken article 45, except that the multiplication and subtraction are here performed actually instead of mentally. For, if we place the numbers thus,

$$\begin{array}{r} 52)11076 \\ \hline 213 \end{array}$$

and say, 52 in 110 twice and 6 over; 52 in 67 once and 15 over; 52 in 156 three times, we have the same quotient; but we find a difficulty in performing the multiplication and subtraction, which would still be increased if the divisor was greater; it is therefore necessary to pursue the actual method.

Note. As there must be (art. 45,) just as many *figures on* the right of the first quotient figure as

there are in the dividend on the right of the part first divided, we easily perceive that when the first partial dividend contains just as many figures as the divisor, the divisor and quotient together will contain one figure more than the dividend; and when the first partial dividend contains one figure more than the divisor, the divisor and quotient together will contain just as many as the dividend. When the dividend is large, we put a dot under each figure as we bring it down, to avoid mistake.

Let the succeding examples be proved both by division and multiplication.

Examples for Practice.

1. $31406 \div 41 = 766$, and $\frac{31406}{766} = 41$. Also, $41 \times 766 = 31406$.
2. $2336 \div 73 = 32$, and $\frac{58716}{63} = 932$.
3. $463088 \div 412 = 1124$, and $\frac{36658994}{514} = 71321$.
4. $7483566 \div 82 = 91263$, and $\frac{27413568}{523} = 52416$.
5. $71710048 \div 796 = 90088$, and $\frac{70469217}{819} = 86043$.

47. The method of finding how often the divisor is contained in each partial dividend, by simply seeking how often its first or left hand figure is contained in the first figure, or two first figures of each of these dividends, is not always infallible; but the multiplication and subtraction which come after always serve to correct the error when there is one. For if, having multiplied by the quotient figure, we find that the product is greater than the dividend from which it is to be subtracted, this is a proof that we have taken the divisor more times than it is contained in this dividend, and consequently, the subtraction is impossible: we must therefore, in this case, diminish the quotient by 1, 2, 3, etc. units till the subtraction becomes possible.

If, on the contrary, having subtracted, we find that the remainder is greater than the divisor, as the divisor is yet contained in the remainder, this is a proof that the quotient figure is too small; it must there-

fore, in this case, be increased till the remainder is less than the divisor.

EXAMPLE.

$170590 \div 473 = 360\frac{310}{473}$.

$$\begin{array}{rl} & \text{Dividend} \\ \text{Divisor } 473) & 170590(360\frac{310}{473} \text{ Quotient} \\ & 1419 \\ \hline & \ \ 2869 \\ & \ \ 2838 \\ \hline & \ \ \ \ 310 \end{array}$$

We here take the four first figures of the dividend because the three first do not contain the divisor.

After which, we seek how often 4 is contained in 17; we find 4 times; but in multiplying the divisor 473 by 4, we have 1892, which is greater than 1705, and consequently the subtraction is impossible; we therefore place only 3 in the quotient. We multiply 473 by 3, and having written the product under 1705, we subtract, and there remains 286, to the side of which we bring down the next figure 9 of the dividend.

We then seek how often 4 is contained in 28; we find 7 times, but for the same reason as above, we place only 6 in the quotient. Having multiplied and subtracted, we have 31 for the remainder, to the side of which we bring down the last figure 0 of the dividend. Then as the divisor is not contained in 310, we place 0 in the quotient. Also, because 310 is a part of the dividend which has not been divided, we place it as a dividend over the divisor as usual, thus $\frac{310}{473}$, and considering this as the quotient of 310 divided by 473, we place it in the quotient on the right hand, as above.

48. The following observation will, in many cases, enable us to avoid useless trials. When the second figure of the divisor (counting from the left) is much

greater than the first, instead of seeking how often the first figure of the divisor is contained in the corresponding part of the dividend, we must seek how often the first figure of the divisor, increased by a unit, is contained in this part. This trial will always be much nearer than the first, and can never give too great a quotient.

Example.

$$\frac{2701}{397}=6\frac{319}{397}.$$

Dividend

$$\text{Divisor } 397)2701(6\frac{319}{397} \text{ Quotient}$$

$$\begin{array}{r} 2382 \\ \hline 319 \end{array}$$

Here, instead of saying in 27 how many times 3, I say in 27 how many times 4, because the divisor is much nearer 400 than 300, I find 6, which is the true quotient; instead of which, by the usual method, I should have found 9, and should consequently have been obliged to make three useless trials. We easily perceive that this can never give too great a quotient; for, if it could, then might 400 be contained in the dividend a greater number of times than 397, which is impossible.

Because $\frac{319}{397}\times 397=319$ (art. 45,) the above example is proved thus: $6\times 397+319=2701$. We shall prove all similar divisions in the same way.

Examples for Practice.

1. $\frac{65497}{296}=221\frac{81}{296}$, and $\frac{728967}{803}=907\frac{646}{803}$.
2. $27065849\div 356=76027\frac{237}{356}$.
3. $\frac{26665942}{886}=30097$, and $\frac{111030011}{199}=557939\frac{150}{199}$.
4. $\frac{3141593}{7854}=399\frac{7847}{7854}$, and $\frac{7847000}{7854}=999\frac{854}{7854}$.
5. $757969331\div 947=800390\frac{1}{947}$.
6. $\frac{3936725}{69}=57053\frac{68}{69}$, and $\frac{12167305}{1219}=9981\frac{466}{1219}$.
7. $78987116249\div 19818=3985624\frac{19817}{19818}$.
8. $\frac{54968\times 42147}{4683}=494712$.

QUESTIONS ON SECTION 6.

1. What does division teach?
2. What is the number called which we divide?
3. What is the number called by which we divide?
4. What is the result of the operation called?
5. Could we find the quotient by subtraction?
6. What sign is placed between two numbers to show that the one is to be divided by the other?
7. When the sign *by* is placed between two numbers, which of the two is the dividend?
8. By what other method do we express the division of one number by another? In this last case, which is the dividend?
9. When the divisor consists of a single figure, and the dividend of many figures, how do we procede?
10. Why do we begin the division at the left hand?
11. Why do we place a cipher in the quotient when any of the partial dividends is too small to contain the divisor?
12. When we speak of numbers and their equimultiples, what is understood?
13. What property belongs to the equimultiples of any two numbers?
14. When the dividend is not an exact multiple of the divisor, how is the division of the remainder expressed, to form a part of the quotient?
15. What is this part of the quotient equal to when multiplied by the divisor?
16. How is a division proved by multiplication when there is a remainder?
17. How is a division proved by another division when there is no remainder?
18. If we divide the product of two numbers by one of them, what will be the result?
19. When the divisor consists of several figures, and we have taken a part of the dividend sufficient

to contain it, how do we find how often it is contained in this part?

20. How do we procede when the second figure of the divisor, counting from the left, greatly excedes the first?

21. What do we observe in comparing the signs *into* and *by*, or the operations of multiplication and division with each other?

Section 7.

49. We will now examine a singular property of the number 9, which furnishes an expeditious method of proving multiplication and division.

If we take all the nines out of a number expressed by any figure followed by ciphers, there will be a remainder of that same figure. This we shall easily perceive, as follows:

$9+1=10$; $99+1=100$; $999+1=1000$, etc.

Therefore, $50=10\times 5$, or $\overline{9+1}\times 5$; $500=100\times 5$, or $\overline{99+1}\times 5$; $5000=1000\times 5$, or $\overline{999+1}\times 5$, etc.

Now, in multiplying 9, 99, 999, etc. by 5, the product will be an exact number of nines; and in multiplying a unit by 5, we shall still have 5; each of the numbers 50, 500, 5000, etc. is therefore composed of an exact number of nines and 5, and consequently, if we take all the nines out of any of these numbers, there will always be a remainder of 5. In the same manner, we might show that each of the numbers 60, 600, 6000, etc. is composed of a certain number of nines and 6; consequently, if we take all the nines out of any of these numbers, there will be a remainder of 6, and it is the same with any other figure followed by ciphers.

Hence, if we take all the nines out of 5647, which is equal to $5000+600+40+7$, the remainder will be the same as when all the nines are taken out of $5+6+4+7$, that is to say, when we take all the nines out of any number, we have the same remain-

der as when we take all the nines out of the sum of the figures which compose that number. Therefore, to find the remainder which we should have in taking all the nines out of 5647, we say, 5 and 6 is 11, and 4 is 15, and 7 is 22. Then, to cast the nines out of 22, we say, 2 and 2 is 4, which is the remainder sought. Or thus: 5 and 6 is 11; cast out 9 and there remains 2: then 2 and 4 is 6, and 7 is 13; cast out 9 and there remains 4, as before.

NOTE. In thus adding we always omit the figure 9.

We prove multiplication by the aid of this property, as follows:

Suppose, that having multiplied 348567 by 562, and found that the product is 195894654, we would prove the correctness of the operation.

We cast the nines out of 348567, and have a remainder of 6: we also cast the nines out of 562, and have a remainder of 4. We multiply these two remainders 6 and 4 together, and casting the nines out of the product 24, there remains 6.

Now if the work is right, in casting the nines out of 195894654, the remainder should be 6, which it is in effect.

These remainders are frequently placed in the angles of a cross, thus:

$$\begin{array}{ccc} & 6 & \\ 6 & \times & 4 \\ & 6 & \end{array}$$

the numbers in the side angles being the remainders after casting the nines out of the two factors; the number in the upper angle the remainder after casting the nines out of the product of the two first remainders, and that in the lower angle the remainder, after casting the nines out of the product, which last will always agree with the number in the upper angle when the work is right.

The reason of this is as follows: 348567 is composed of a certain number of nines and a remainder of 6, and 562 is composed of a certain number of nines and a remainder of 4: now the nines of the two factors multiplied together will produce nines; *also, these* nines multiplied by either of the remain-

ders 4 and 6, will produce nines: hence the product of 6 and 4 is the only part of the total product which is not exactly divisible by 9; consequently, the remainder, in casting the nines out of the product of these two remainders, must be the same as the remainder in casting the nines out of the total product.

The same property appertains to the number 3, in consequence of its being a measure of 9.

To prove division in the same manner, we have only to recollect that the dividend is the product of the divisor and quotient. When the division has left a remainder, we cast the nines out of it, and add the remainder to the figure in the upper angle, casting the nines out of the sum when it excedes nine.

This proof is not absolute; for if in the operation there should be two errors which balance, or if there should be an error of nine, or any number of nines, as that would not alter the remainder in casting out the nines, we should not discover the mistake. But either of these cases is very improbable, and therefore will be very rare in practice. Hence the proof by 9 is esteemed a very useful verification.

For practice, let the examples (art's. 35 and 48,) be proved by casting out the nines.

50. If a number will divide the whole of another number, and one of its parts, it will also divide the other part. Now 4 divides 32, and because $32=4+4+4+4+4+4+4+4$, if we take from 32 a part which is a certain number of fours, it is plain that the remainder will also be a certain number of fours. Therefore, if a number, etc.

51. We have seen (art. 41,) that $(8+6+3)4=32+24+12$, that is to say, that 68, the product of 17×4, is the same as the sum of the products when all the parts of 17 are multipled separately by 4; consequently, the quotient 17 of $68\div4$ will be the same as the sum of the quotients when all the parts of 68 are separately divided by 4, that is to say, $\frac{32}{4}+\frac{24}{4}+\frac{12}{4}=8+6+3=17$;

or, $\frac{32}{4}+\frac{24}{4}+\frac{12}{4}=\frac{32+24+12}{4}=\frac{68}{4}=17$; and this is general, whatever be the number of parts or the divisor. Hence, also, it is evident, that if a number divides all the parts of another number, it will also divide the whole of that number.

Therefore, if we divide the sum of several numbers by any number, the quotient will equal the sum of the quotients, when the numbers are separately divided by that same number; consequently, when we have several numbers, to divide each by the same number, we may add them together, and divide their sum by that number.

Thus, $\frac{5}{4}+\frac{1}{4}+\frac{3}{4}=\frac{5+1+3}{4}=\frac{9}{4}=2\frac{1}{4}$.

TO FIND THE GREATEST COMMON MEASURE OF TWO NUMBERS.

52. A number is called a *measure* or *aliquot part* of its multiple: thus 3 is a measure of 12. Also, a number is a *common measure* of all its multiples: thus 3 is a common measure of 6, 9, 12, 15, 18, etc.

A number which is not divisible by any number but itself or a unit is a *prime number:* thus, 2, 3, 5, 7, 11, 13, 17, 19, 23, etc. are prime numbers. Also, those numbers, whether primes or not, which have no common measure greater than a unit, are prime to each other: for example, 7 and 8 are prime to each other.

We will now show the method of finding the greatest common measure of two numbers which are not prime to each other, which method also discovers whether two numbers are prime to each other or not. This method is as follows:

First divide the greater number by the less, and if there is no remainder the number divided by is the greatest common measure.

If there is a remainder, divide the less number by *this remainder*, and if this division gives no remain-

der, the number last divided by is the number sought.

If there is still a remainder, divide the first remainder by the second; and thus we continue always dividing the preceding remainder by the last, till the division becomes exact.

The number last divided by will be the greatest common measure.

If the last divisor is a unit, the numbers are prime to each other.

NOTE. Numbers which are not primes are called *composite numbers.*

EXAMPLES.

1. Find the greatest common measure of the numbers 4446 and 6498, as directed above, thus:

```
4446)6498(1
     4446
     ----
     2052)4446(2
          4104
          ----
           342)2052(6
               2052
               ----
               ****
```

The last divisor, 342, is the number sought.

As 342 divides 2052, it will also divide 4104, which is twice 2052; but if a number divides all the parts of another it will also divide the whole, (art. 51,) therefore 342 divides 4446, which is the sum of 4104+342; again, because 342 divides 2052 and 4446, it also divides their sum, which is 6498, wherefore 342 is a common measure of 4446 and 6498.

No number greater than 342 can be a common measure, for, if possible, let 343 be a common measure.

Then, because 343 divides 6498 and 4446, it will also divide 2052, their difference, seeing that a num-

ber which divides the whole of another, and one of its parts, will also divide the other part.

Again, because 343 divides 2052, it will also divide 4104, which is twice 2052.

Lastly, because 343 divides 4446 and 4104, and because a number which divides the whole of another and one of its parts will also divide the other part, (art. 50,) 343 will divide their difference 342, which is absurd, therefore 343 is not a common measure. In the same manner it may be proved that no other number greater than 342 can be a common measure of 4446 and 6498. Wherefore 342 is the greatest common measure.

2. Required the greatest common measure of 3675 and 5880. Answer, 735.

3. Find the greatest common measure of 23205 and 31395. Answer, 1365.

4. Required the greatest common measure of 4437 and 5899. Answer, 17.

5. Required the greatest common measure of 46503 and 57546. Answer, 9.

TO FIND THE LEAST COMMON MULTIPLE OF SEVERAL NUMBERS.

53. When the numbers are prime to each other, multiply them all together, and the product is the least common multiple.

When the numbers are not prime to each other, draw a line underneath, and divide two or more of them by the least prime number by which this can be done, placing those numbers which cannot be divided below the line, together with the quotients. Divide two or more of the numbers below the line by the same divisor as often as this can be done. Having exhausted the first divisor, take for a divisor the next least prime that will divide two or more, with which procede as with the first. Continue thus till the quotients and undivided numbers, that is to say, *the numbers* below the line, are prime to each other.

Lastly, multiply the divisors and remaining numbers together, and the product is the least common multiple.

Example.

To find the least common multiple of 2, 6, 7, and 14.

I place the numbers thus:

$$\begin{array}{l} 2)\underline{2,\ 6,\ 7,\ 14} \\ 7)\underline{1,\ 3,\ 7,\ \ 7} \\ \quad 1,\ 3,\ 1,\ \ 1 \qquad 2\times7\times3=42 \end{array}$$

and having found that 2 will divide 2, 6, and 14, I divide by 2, and have the quotients 1, 3, and 7, which I place respectively under the figures divided: also, as 2 will not divide 7, I place 7 under the line, together with the quotients found. Now it is plain that no two of the numbers 1, 3, 7, 7, can be divided by 2, 3, or 5; but as there are two sevens, I divide by 7, and have the suite, 1, 3, 1, 1, which can no longer be divided. Lastly, I multiply the two divisors and the remaining number 3, which gives 42, for the least common multiple of the numbers 2, 6, 7, and 14. The three quotients 1, 1, 1, are omitted because 1 neither multiplies nor divides.

That the number found by this method will always be the least common multiple of the given numbers, will appear from the following illustration:

If in the above example we multiply the first line of quotients, 1, 3, 7, by the divisor 2, we shall reproduce the numbers divided; also, because the 7 which was not divided is retained, the product of the divisor 2 and the numbers 1, 3, 7, 7, will be a multiple of all the given numbers. Again, for the same reason, the product of the last divisor 7 and the numbers 1, 3, 1, 1, which is 21, will be a multiple of the numbers 1, 3, 7, 7; also, (art. 43, ex. 10,)

it is evident that if 21 is a multiple of any number, twice 21 will be the same multiple of twice that number. Therefore twice 21, or 42, is a common multiple of the given numbers 2, 6, 7, and 14.

Also 42 is the least common multiple of the numbers 2, 6, 7, 14, for, if not, let 41 be the least common multiple: then, by this supposition, 14 divides 41. Now $41=28+13$; then, because 14 divides 41 and the part 28, and because any number which divides the whole of another and one of its parts will also divide the other part, 14 will divide 13, which is absurd. Therefore 41 is not a common multiple of the numbers 2, 6, 7, and 14; and in the same manner it may be proved that no number less than 42 can be a common multiple of these numbers; wherefore 42 is the least common multiple.

The reason why we divide by the prime numbers, as directed by the rule, will appear from the following example:

Required the least common multiple of 3, 6, 9, 27, and 54.

If we procede according to the rule, we find 54 for least common multiple; but if we take composite numbers to divide by, thus:

$$\begin{array}{r|l} 9) & 3,\ 6,\ 9,\ 27,\ 54 \\ \hline 6) & 3,\ 6,\ 1,\ \ 3,\ \ 6 \\ \hline 3) & 3,\ 1,\ 1,\ \ 3,\ \ 1 \\ \hline & 1,\ 1,\ 1,\ \ 1,\ \ 1 \end{array} \qquad 9\times6\times3=162$$

we find 162, which is a common multiple, but not the least.

Let us here observe that in dividing by 9 instead of dividing by 3, and again by 3, we lose the reduction of those numbers which are less than 9, and which are divisible by 3, which is the reason that by this method we do not obtain the least common multiple.

Examples for Practice.

1. Required the least common multiple of 3, 5, 7, 4, and 11. Answer, 4620.

2. What is the least common multiple of 2, 3, 4, 6, 7, 12, 14, 21, 28, and 42? Answer, 84.

3. What is the least common multiple of 2, 5, 10, 11, and 55? Answer, 110.

4. What is the least common multiple of 8, 9, 12, 15, 24, and 27? Answer, 1080.

5. What is the least common mutiple of the nine digits? Answer, 2520.

54. Let us again observe that the two operations of multiplication and division destroy the effect of each other. Thus $\frac{24\times4}{4}=24$; therefore whenever we have a number to multiply and divide by the same number, we may spare ourselves the trouble of both operations.

Also, because (art. 43) the equimultiples of any two numbers contain each other as often as the numbers contain each other, whenever we have to multiply a dividend and divisor both by the same number, as this will have no effect upon the quotient, we may omit such multiplication; thus $\frac{24\times4}{8\times4}=\frac{24}{8}=3$.

55. Though many changes may be made in the form under which a number is represented, yet no change can be made in its value, without increasing or diminishing it. Now we cannot increase a number but by *addition* or *multiplication*, neither can we diminish it but by *subtraction* or *division*; therefore every arithmetical calculation requires the exercise of some or all of these operations, for which reason they are styled the fundamental operations of arithmetic.

Questions on Section 7.

1. What peculiarity belongs to the number nine?
2. How do we apply this to prove multiplication and division?
3. Is this proof infallible, and if not, why is it considered valuable as a verification?
4. How do we show that if a number divides another and one of its parts, it will also divide the other part?
5. What do we observe in multiplying or dividing a number by another, and in multiplying or dividing all its parts separately by that other number?
6. What advantage do we derive from this principle?
7. What is meant when we say that one number is a measure or aliquot part of another?
8. What is a common measure?
9. What is a prime number?
10. What is a composite number?
11. When are numbers prime to each other?
12. How do we procede in order to find the greatest common measure of two given numbers? Explain the nature of this operation by an example.
13. What will the last divisor be when the numbers are prime to each other?
14. How do we find the least common multiple of several numbers which are prime to each other?
15. How do we procede when the numbers are not prime? Let the nature of this operation be explained by an example.
16. Why do we take prime numbers for divisors in preference to composite numbers?
17. Why do we omit the multiplication of a dividend and divisor by the same number?
18. Why are addition, subtraction, multiplication,

and division, called the fundamental operations of arithmetic?

SECTION 8.

FRACTIONS.

56. When a single unit of any kind is divided into any number of equal parts, we call any number of these parts less than the whole *a fraction*. Also any possible quantity less than a unit is called *a fraction*.

For example, suppose that to measure a quantity of grain we choose a bushel for the *unit* or *measure*, and that, having found 20 bushels in the quantity, there is at last a remainder not sufficient to fill the bushel. This remainder is *a fraction* of a bushel.

To estimate this fraction of a bushel, we divide the bushel or measure first taken into a number of equal parts, suppose 4: one of these parts we call a peck; then suppose that, taking a peck for the *unit*, we measure the remaining quantity or fraction of a bushel with this peck, and find that it contains the peck just 3 times: we then say that the whole quantity of grain contains 20 bushels and 3 pecks. Now the fraction 3 pecks is just as much a whole number of pecks as 20 bushels is a whole number of bushels; but when we wish to give the denomination of bushel to this fraction, we write it thus, $\frac{3}{4}$, and read *three-fourths* of a bushel, the number under the line being the number of equal parts or smaller units into which the bushel is divided, and the number above the line being the number of these units or parts contained in the fraction. Therefore 20 bushels 3 pecks is 20 bushels and $\frac{3}{4}$ of a bushel, or $20\frac{3}{4}$ bushels.

Again, suppose that, having found 20 bushels 3 pecks in the quantity of grain above mentioned, there is still a remainder not sufficient to fill a peck,

and that, wishing to estimate this fraction of a peck, we divide the peck into 8 equal parts, one of which we call a quart; then, taking a quart for the *unit*, suppose that this unit is contained in the above fraction of a peck 7 times, the whole quantity of grain then contains 20 bushels 3 pecks 7 quarts. As the quantity 7 quarts is less than a peck, and as it would require 4 pecks to make a bushel, the quantity 3 pecks 7 quarts is therefore less than a bushel, that is to say, it is *a fraction* of a bushel. Now to give the denomination of bushel to this fraction 3 pecks 7 quarts, we reduce it all to quarts, to do which, because 8 quarts make 1 peck, we say, $3\times8+7=31$; then placing 31 over 32, the number of quarts in a bushel, we have $\frac{31}{32}$ for the fraction of a bushel equivalent to 3 pecks 7 quarts. Therefore the quantity 20 bushels 3 pecks 7 quarts is equal to $20\frac{31}{32}$ bushels, that is to say, to 20 bushels and 31 of the parts of which a bushel contains 32. Hence we see that the units of which a fraction is composed are frequently represented by a whole number, their species being in this case determined by a particular name. We shall consider them under this form more particularly hereafter.

The fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, are read *one-half*, *two-thirds*, *three-fourths*, *five-sixths*, *seven-eighths*, *nine-tenths*, and other fractions in like manner.

57. The two numbers which constitute a fraction are called the *terms* of the fraction. The term above the line is called *numerator*, and that below the line, *denominator*.

Fractions considered as parts of a unit, are sometimes called *broken numbers*, in contradistinction to whole units, which are also sometimes called *integers*. A number which consists of a whole number and a fraction is called a *mixed number*. Thus $5\frac{3}{4}$ is a mixed number, of which we say that 5 is the integral part and $\frac{3}{4}$ the fractional part.

Whole and mixed numbers are frequently represented under the form of fractions, as we have seen

in division, in which case they are called *improper fractions.*

58. When we consider a fraction as representing a certain part of a unit, we conceive that the unit is divided into as many equal parts or smaller units as are signified by the denominator of the fraction, and that the value of the fraction contains as many of these smaller units as are signified by its numerator.

We easily perceive that the number of parts into which the unit is divided must always determine the value of these parts, that is to say, that these parts must be greater in proportion as their number is less, and on the contrary, less in proportion as their number is greater; and therefore if we make the number of these two, three, four, etc. times greater, there must be two, three, four, etc. times as many of them to compose the same quantity.

Let us take, for example, the fraction $\frac{1}{3}$, and consider it as representing one-third of a shilling; then, as the shilling is 12 pence, and as by the denominator 3 we conceive this shilling to be divided into 3 equal parts, the value of each part is 4 pence. Now if we multiply the denominator 3 by 2, we shall have $\frac{1}{6}$, that is to say, twice as many parts in the shilling; and, as the value of each of these is only 2 pence, we must take 2 of them to have the value of $\frac{1}{3}$, which is 4 pence, that is to say, we must multiply the numerator by 2 as well as the denominator, in order that the fraction may still retain the same value. Thus we see that $\frac{1}{3}=\frac{2}{6}=\frac{4}{12}=\frac{8}{24}=\frac{16}{48}$, etc., that is to say, *that the value of a fraction is not altered when both its terms are multiplied by the same number.* Also, because $\frac{16}{48}=\frac{8}{24}=\frac{4}{12}=\frac{2}{6}=\frac{1}{3}$, *the value of a fraction is not altered when both its terms are divided by the same number.* Hence the same fraction may be expressed by an infinity of different numbers.

Again, we may consider the numerator of a fraction as a whole number which is to be divided by the denominator. For example, if we take any of the above values of $\frac{1}{3}$, and divide the numerator

considered as a number of shillings by the denominator, we shall still have 4 pence for the result. Thus, if we take $\frac{8}{24}$, because a shilling is equal to 12 pence, 8 shillings are equal to 96 pence, and $\frac{96}{24}=4$, the value of $\frac{1}{3}$ of a shilling, in pence, as before. Hence we discover that a twenty-fourth part of 8 shillings is the same as 8 twenty-fourth parts of one shilling. The fractions $\frac{7}{8}$, $\frac{17}{21}$, $\frac{43}{64}$, may therefore be read, *seven divided by eight*, *seventeen divided by twenty-one*, *forty-three divided by sixty-four*, and all other fractions, whether proper or improper, in like manner.

59. As a whole number divided by a unit is still the same, we may always express a whole number fractionally by giving it a unit for a denominator: thus $5=\frac{5}{1}$, $6=\frac{6}{1}$, etc. Also, because a number is not altered (art. 44) when it is multiplied and divided by the same number, we may give a whole number what denominator we please by first multiplying it by that denominator. For example, if we would express 9 fractionally, and give it 8 for a denominator, we first multiply 9 by 8, which gives 72 for the numerator, under which we place the denominator 8, thus $\frac{72}{8}$.

60. To bring a mixed number to the form of an improper fraction, we reduce the integral part to a fraction, having the same denominator as the fractional part, after which we add the integral and fractional parts together. For example, if we would express $5\frac{3}{4}$ by its equivalent improper fraction, we reduce 5 to an improper fraction, having 4 for its denominator, thus $\frac{5\times 4}{4}=\frac{20}{4}$; then (art. 51) we have $\frac{20}{4}+\frac{3}{4}=\frac{23}{4}$, which is the fraction sought.

Or thus, considering $5\frac{3}{4}$ as the quotient of a division where the divisor was 4, we shall multiply this quotient by the divisor 4, which (art. 43,) will give the dividend. Now, (art. 44,) because $\frac{3}{4}\times 4=3$, we say, 4 times 5 is 20, and 3 is 23, which is the dividend; therefore, in placing the divisor underneath, *thus*, $\frac{23}{4}$, we have the fraction sought. The same

will apply to any other mixed number, that is to say, we may consider every mixed number as the result of a divison, the divisor of which is expressed by the denominator of its fractional part. Let us observe, that $5\frac{3}{4}$ is the same as $5+\frac{3}{4}$.

EXAMPLES.

1. Required the improper fraction equivalent to $7+\frac{7}{8}$.

I procede thus: $7=\frac{7\times 8}{8}=\frac{56}{8}$, (art. 59,) consequently, $7+\frac{7}{8}=\frac{56}{8}+\frac{7}{8}$; but (art. 51,) $\frac{56}{8}+\frac{7}{8}=\frac{56+7}{8}=\frac{63}{8}$, and therefore, $7+\frac{7}{8}=\frac{63}{8}$.

Let the mixed numbers in the following example, be expanded in the same manner.

2. $9\frac{2}{3}=\frac{29}{3}$; $11\frac{6}{7}=\frac{83}{7}$; $15\frac{4}{11}=\frac{169}{11}$, and $34\frac{5}{13}=\frac{447}{13}$.

Now if the young student has not expanded these numbers as in the preceding example, I advise him to do it before he procedes farther, and also to refer to the articles 51 and 59.

3. $5\frac{1}{6}=\frac{5\times 6+1}{6}=\frac{31}{6}$, and $13\frac{2}{5}=\frac{13\times 5+2}{5}=\frac{67}{5}$.

4. $8\frac{8}{9}=\frac{80}{9}$; $25\frac{2}{7}=\frac{177}{7}$; $106\frac{3}{10}=\frac{1063}{10}$, and $531\frac{11}{12}=\frac{6383}{12}$.

5. $98\frac{13}{14}=\frac{1385}{18}$; $34\frac{19}{41}=\frac{1413}{41}$, and $504\frac{27}{40}=\frac{20187}{40}$.

61. As the denominator of a fraction determines the number of parts into which the unit is divided, and the numerator the number of these parts signified by the fraction; it is evident, (the denominator remaining the same,) that if we take 2, 3, etc. times the numerator, we shall have 2, 3, etc. times the value of the fraction; and on the contrary, if we take one-half, one-third, etc. of the numerator, we shall have one-half, one-third, etc. of the value of the fraction.

Again, (the numerator remaining the same,) if we

multiply the denominator by 2, 3, etc. as this to conceive 2, 3, etc. times as many parts in the unit, the numerator will then signify a number of parts 2, 3, etc. times less than the first; and consequently, the fraction will signify only one-half, one-third, etc. of its former value: on the contrary, if we divide the denominator of a fraction by 2, 3, etc. (the numerator remaining still the same,) as the value of the parts contained in the unit will be 2, 3, etc. times greater than before, the fraction will be 2, 3, etc. times greater. Hence the following general rule.

To multiply a fraction, *multiply its numerator, or divide its denominator;* and to divide a fraction, *divide its numerator, or multiply its denominator.*

NOTE. The multiplication, or division of a fraction by division, is not always practicable; but when practicable, it is the most concise method.

EXAMPLES.

1. $\frac{3}{4}\times 4=\frac{3\times 4}{4}=\frac{12}{4}=3$, or $\frac{3}{4}\times 4=\frac{3}{4\div 4}=\frac{3}{1}=3$.

As $4=\frac{4}{1}$, the expression $\frac{3}{4}\times 4$ is the same as $\frac{3}{4}\times\frac{4}{1}$. Hence we see that the product of two fractions is found by multiplying their numerators together for a numerator, and their denominators together for a denominator.

2. $\frac{5}{6}\times 3=\frac{5}{6}\times\frac{3}{1}=\frac{15}{6}=\frac{5}{2}=2\frac{1}{2}$;

or, $\frac{5}{6}\times 3=\frac{5}{6\div 3}=\frac{5}{2}=2\frac{1}{2}$.

3. $\frac{6}{7}\div 3=\frac{6\div 3}{7}=\frac{2}{7}$, or, $\frac{6}{7}\div 3=\frac{6}{7\times 3}=\frac{6}{21}=\frac{2}{7}$.

4. $\frac{8}{9}\div 4=\frac{8\div 4}{9}=\frac{2}{9}$, or, $\frac{8}{9}\div 4=\frac{8}{9\times 4}=\frac{8}{36}=\frac{2}{9}$.

The expression $\frac{5}{6\div 3}$ is the same as $\frac{5}{\frac{6}{3}}$, and as we *have seen* (ex. 2,) that this is of the same value as $\frac{15}{6}$, *we learn* that when we have a whole number to di-

vide by a fraction, we can express the value or quotient by a simple fraction, by multiplying the whole number by the denominator of the fraction for a numerator, leaving the numerator of the fraction for the denominator. Thus $\frac{7}{\frac{8}{9}}=\frac{63}{8}$; $\frac{9}{\frac{2}{3}}=\frac{27}{2}$; $\frac{13}{\frac{5}{8}}=\frac{104}{5}$.

TO REDUCE FRACTIONS TO THEIR LOWEST TERMS.

62. A fraction is said to be expressed in its lowest terms when these terms are prime to each other. Thus $\frac{1}{3}$, $\frac{5}{7}$, $\frac{9}{10}$, $\frac{13}{25}$, are fractions expressed in their lowest terms; but this is evidently not the case with $\frac{2}{6}$, $\frac{4}{12}$, $\frac{8}{24}$, each of which is equal to $\frac{1}{3}$.

When the terms in which a fraction is expressed are not its lowest terms, we find the latter in dividing the former by their greatest common measure. For example, to find the lowest terms of the fraction $\frac{6209}{11531}$, I first find the greatest common measure of 6209 and 11531, (art. 52,) which is 887, I then divide each term of the fraction by 887, which (art. 58,) does not alter its value, and gives $\frac{7}{13}$ for the result.

EXAMPLES FOR PRACTICE.

1. 65) $\frac{195}{455}=\frac{3}{7}$; $\frac{580}{928}=\frac{5}{8}$; $\frac{714}{1785}=\frac{2}{5}$, and $\frac{108}{684}=\frac{3}{19}$.
2. $\frac{3675}{5880}=\frac{5}{8}$; $\frac{4437}{5899}=\frac{261}{347}$, and $\frac{56837}{69344}=\frac{5167}{6304}$.
3. $\frac{12989}{13827}=\frac{31}{33}$; $\frac{13549}{23113}=\frac{17}{29}$, and $\frac{6978}{5343}=$ ———.
4. $\frac{14697}{83766}=\frac{213}{1214}$, and $\frac{73485}{235152}=\frac{5}{16}$.

In addition to what has been said, we may observe, that any even number is divisible by 2; any number, of which the sum of the figures is a multiple of 9 or of 3, is divisible by 9 or by 3, (see proof by 9, art. 49,) and any number having 5 or a cipher in the place of units, is divisible by 5. See Pythagorean Table.)

Therefore, when the two terms of a fraction are even numbers, we may divide them both by 2; when *the sum of* the figures of each is divisible by 9 or by

3, we may divide both by 9 or by 3, accordingly; when the unit figure of each is 5, or when the unit figure of the one is 5, and of the other a cipher, we may divide both by 5; when both terminate with ciphers, we may cut off an equal number of ciphers from each, as this will still be dividing both by the same number; and lastly, when the terms are nearly equal to each other, we may try to divide both by their difference, for if this succedes, it is the greatest number that will divide both without a remainder; and if any number will divide both terms without a remainder, their difference is either that number, or a multiple of that number. This last we shall illustrate thus:—Suppose that when a fraction is at its lowest terms, the difference between the terms is a unit, then if we multiply both terms by the same number, the difference between the products will be the number by which we multiply, and will consequently be the greatest number by which both products can be divided, otherwise the fraction could not have been expressed in its lowest terms.

Again, when the difference between the lowest terms of a fraction is several units, if we multiply both by the same number, it is evident that the difference of the products will be as many times the number by which we multiply as there is a unit in the difference between the terms; therefore, the difference between the products will be a multiple of the number by which both terms were multiplied, and consequently, a greater number than will divide both.

If we would reduce $\frac{12989}{13827}$ to its lowest terms, I subtract the numerator from the denominator, and I have 838 for the difference. Now it is easy to perceive, that 838 will divide neither of the terms, because no whole number multiplied by the unit figure 8 will produce either 9 or 7. But if any number will divide both terms, 838 must be a multiple of that number; I therefore divide 838 by 2, and I have 419, by which number I try to divide both terms; the

division succedes, and gives me $\frac{31}{35}$ for the terms sought.

Yet notwithstanding what we have here said, and all that might farther be said upon this subject, of all methods for finding the least terms of a fraction, that of dividing both terms by their greatest common measure is the most prompt and certain.

TO REDUCE FRACTIONS HAVING DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING A COMMON DENOMINATOR.

63. When the denominators are prime to each other, we multiply them together, and take their product, which is their least common multiple, for a common denominator.

For the new numerators, we take each numerator as often as its denominator is contained in the common denominator; or, which is the same thing, we multiply each numerator into all the denominators except its own, and placing the several products over the common denominator, we have the fractions sought.

For example, if we have $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{3}{5}$ to reduce to equivalent fractions having a common denominator, we procede thus:

$3 \times 7 \times 5 = 105$ common denominator.
Then $2 \times 7 \times 5 = 70$ new numerator for $\frac{2}{3}$.
$5 \times 3 \times 5 = 75$ new numerator for $\frac{5}{7}$.
$3 \times 7 \times 3 = 63$ new numerator for $\frac{3}{5}$.

and placing each new numerator over the common denominator, we have $\frac{70}{105}$, $\frac{75}{105}$, and $\frac{63}{105}$ for the fractions required.

In multiplying the three denominators together, we have evidently multiplied the denominator of each fraction by the product of the denominators of the other two; but we have likewise multiplied the numerator of each fraction by the same product. We have therefore multiplied the numerator and de-

nominator of each fraction by the same number, which (art. 58,) does not alter its value.

Therefore, the fractions $\frac{70}{105}$, $\frac{75}{105}$, and $\frac{63}{105}$, are respectively equal to the fractions $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{3}{5}$, and they all have the same denominator as was required.

EXAMPLES.

1. $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{3}{5}$ are equivalent to $\frac{15}{60}$, $\frac{40}{60}$, and $\frac{36}{60}$.

2. $\frac{4}{5}$, $\frac{4}{9}$, and $\frac{6}{7}$, are equivalent to $\frac{252}{315}$, $\frac{140}{315}$, and $\frac{270}{315}$.

3. $\frac{3}{4}$, $\frac{5}{9}$, $\frac{6}{7}$, and $\frac{7}{11}$, are equal to $\frac{2079}{2772}$, $\frac{1540}{2772}$, $\frac{2376}{2772}$, and $\frac{1764}{2772}$.

4. $\frac{3}{8}$, $\frac{5}{13}$, and $\frac{8}{15}$, are equal to $\frac{585}{1560}$, $\frac{600}{1560}$, and $\frac{832}{1560}$.

5. $\frac{7}{19}$, $\frac{9}{10}$, and $\frac{13}{23}$, are equal to $\frac{1610}{4370}$, $\frac{3933}{4370}$, and $\frac{2470}{4370}$.

64. When the denominators are not prime to each other, we find their least common multiple, which we take for a common denominator. After which, having placed the fractions by the side of each other, and drawn a line underneath, we divide the common denominator by the several denominators of the given fractions, placing each quotient below the line, and under the denominator which gave it. Lastly, we multiply the two terms of each fraction by the corresponding number below the line, and we have the fractions sought.

NOTE. We need not multiply each denominator by the number below the line, because we already know the result.

For example, to reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{5}{16}$ to a common denominator, we place the fractions thus:

$$\begin{array}{cccc} \frac{2}{3}, & \frac{3}{4}, & \frac{7}{8}, & \frac{5}{16} \\ \hline 16 & 12 & 6 & 3 \end{array}$$

and having found (art. 53,) that the least common multiple of the several denominators is 48, we take this number for a common denominator. We then seek how often the denominator of each fraction is contained in 48, the common denominator, and having

found the quotients 16, 12, 6, and 3, we place each of these under the denominator which gave it. Lastly, multiplying the two terms of each fraction by the corresponding number below the line, we have $\frac{32}{48}$, $\frac{36}{48}$, $\frac{42}{48}$, and $\frac{15}{48}$ for the required fractions.

Because we have multiplied the two terms of each fraction by the same number, the value of each is not altered, (art. 58.) Therefore, the fractions $\frac{32}{48}$, $\frac{36}{48}$, $\frac{42}{48}$, and $\frac{15}{48}$ are equivalent to the given fractions, and they have a common denominator, as was required.

As no number less than 48 will contain each of the denominators a certain number of times, it is evident that the least common multiple will in all cases be the least possible common denominator.

By the method given in the preceding article, we should have had 1536 for the common denominator, and consequently $\frac{1024}{1536}$, $\frac{1152}{1536}$, $\frac{1344}{1536}$, and $\frac{480}{1536}$, for the required fractions, which indeed are of the same value as the former, but would be found much less commodious in calculation. Hence we easily perceive the advantage of finding the least common denominator.

EXAMPLES.

1. $\frac{5}{6}$, $\frac{11}{12}$, and $\frac{7}{15}$, are equal to $\frac{50}{60}$, $\frac{55}{60}$, and $\frac{28}{60}$.

2. $\frac{5}{8}$, $\frac{7}{9}$, $\frac{35}{36}$, and $\frac{11}{24}$, are equal to $\frac{45}{72}$, $\frac{56}{72}$, $\frac{70}{72}$, and $\frac{33}{72}$.

3. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, and $\frac{7}{12}$, are equal to $\frac{16}{24}$, $\frac{18}{24}$, $\frac{4}{24}$, $\frac{3}{24}$, and $\frac{14}{24}$.

4. $\frac{5}{6}$, $\frac{8}{9}$, $\frac{17}{18}$, $\frac{26}{27}$, and $\frac{37}{54}$, are equal to $\frac{45}{54}$, $\frac{48}{54}$, $\frac{51}{54}$, $\frac{52}{54}$, and $\frac{37}{54}$.

5. $\frac{2}{3}$, $\frac{16}{17}$, $\frac{19}{24}$, and $\frac{31}{51}$, are equal to $\frac{272}{408}$, $\frac{384}{408}$, $\frac{323}{408}$, and $\frac{248}{408}$.

QUESTIONS ON SECTION 8.

1. What is a fraction?
2. *How* is a fraction expressed?

3. Are the units composing a fraction sometimes represented by a whole number? Give an example.

4. What general name do we give the two numbers which constitute a fraction?

5. What is the particular name of each term?

6. Why are fractions called broken numbers?

7. What is meant by the word integer?

8. What is a mixed number?

9. What name do we give to a whole or mixed number when represented under the form of a fraction?

10. Why does the value of a fraction still remain the same when its terms are both multiplied, or both divided by the same number?

11. By what different methods can we estimate the fraction $\frac{3}{5}$ of a dollar, and in what other way may this and other fractions be read?

12. How may the same fraction be represented by different numbers?

13. What general denominator do we give to whole numbers in representing them as fractions?

14. In what way do we represent a whole number by a fraction, having any denominator that we please?

15. How do we bring a mixed number to an improper fraction?

16. What is the general method of multiplying a fraction by a whole number? By what other method can we sometimes do this?

17. When both these methods are practicable, which is the most concise?

18. What is the general method of dividing a fraction by a whole number, and when can this be done by division?

19. When this can be done by either method, which do you prefer, and why?

20. How is a whole number divided by a fraction?

21. When is a fraction said to be expressed in its *lowest terms*?

22. When the terms of a fraction are not prime to each other, how do we reduce this fraction to its lowest terms?

23. How do we reduce fractions having different denominators to equivalent fractions having a common denominator, supposing the denominators to be prime numbers, or prime to each other?

24. How do we procede when the denominators are not prime to each other?

SECTION 9.

ADDITION OF FRACTIONS.

65. When the fractions to be added have a common denominator, we add all the numerators, and place the sum over the common denominator. The fraction thus formed is (art. 51,) the sum of the given fractions. Thus $\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$.

But when they have not a common denominator, we must first reduce them to a common denominator, (art. 63 or 64,) because (art. 25,) we cannot add units of a different kind to each other, after which we procede as before. For example, to find the sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{7}$, I first reduce these fractions (art. 63,) to a common denominator, and I have $\frac{21}{42}$, $\frac{28}{42}$, and $\frac{18}{42}$. Then (art. 51,) to find their sum, I say, $\frac{21}{42}+\frac{28}{42}+\frac{18}{42}=\frac{21+28+18}{42}=\frac{67}{42}=1\frac{25}{42}$. Wherefore, the sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{7}$, is $1\frac{25}{42}$.

EXAMPLES.

1. $\frac{3}{4}+\frac{5}{9}+\frac{6}{7}=\frac{189}{252}+\frac{140}{252}+\frac{216}{252}=\frac{189+140+216}{252}$
$=\frac{545}{252}=2\frac{41}{252}$.

Hence the sum of $\frac{3}{4}$, $\frac{5}{9}$, and $\frac{6}{7}$, is $2\frac{41}{252}$. Let the expressions in the following examples be expanded in the same manner.

2. $\frac{3}{5}+\frac{4}{7}+\frac{5}{11}=\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d}=\frac{626}{385}=1\frac{241}{385}$.
The letters a, b, c, represent the new numerators, and d the common denominator.

3. (Art. 64.) $\frac{2}{3}+\frac{5}{6}+\frac{7}{8}=\frac{16}{24}+\frac{20}{24}+\frac{21}{24}=\frac{16+20+21}{24}$ $=\frac{57}{24}=\frac{19}{8}=2\frac{3}{8}$.

4. $\frac{1}{6}+\frac{6}{7}+\frac{13}{14}+\frac{1}{21}=\frac{a}{d}+\frac{b}{d}+\frac{c}{d}+\frac{e}{d}=\frac{a+b+c+e}{d}=2$.

5. $\frac{1}{9}+\frac{2}{7}+\frac{5}{21}+\frac{16}{63}=\frac{8}{9}$ (art. 62.) Also, $\frac{1}{11}+\frac{2}{5}+\frac{28}{55}=1$.

6. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{7}{8}+\frac{8}{9}=5\frac{113}{360}$.

7. $\frac{10}{11}+\frac{5}{12}+\frac{9}{13}+\frac{16}{17}=2\frac{27983}{29172}$.

8. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}=1\frac{2089}{2520}$.

ADDITION OF MIXED NUMBERS.

66. Mixed numbers may be added either by first changing them to improper fractions, and then proceding as in the above article, or by first finding the sum of the integral parts, then that of the fractional parts, and lastly, the sum of these two sums.

Examples.

1. To add $5\frac{1}{2}$, $6\frac{3}{4}$, and $7\frac{5}{8}$ together, I procede thus: $5\frac{1}{2}=\frac{11}{2}$; $6\frac{3}{4}=\frac{27}{4}$, and $7\frac{5}{8}=\frac{61}{8}$, (art. 60;) therefore, $5\frac{1}{2}+6\frac{3}{4}+7\frac{5}{8}=\frac{11}{2}+\frac{27}{4}+\frac{61}{8}=\frac{44}{8}+\frac{54}{8}+\frac{61}{8}$ $=\frac{44+54+61}{8}=\frac{159}{8}=19\frac{7}{8}$. Or thus, $5+6+7=18$, the sum of the integral parts: then $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}=\frac{4}{8}+\frac{6}{8}$ $+\frac{5}{8}=\frac{4+6+5}{8}=\frac{15}{8}=1\frac{7}{8}$, the sum of the fractional parts; and lastly, $18+1\frac{7}{8}=19\frac{7}{8}$, the sum total, as before.

The student may pursue whichever of these two methods seems most convenient according to circumstances. The one is also a very good proof of the other.

2. $2\frac{1}{2}+1\frac{1}{9}+3\frac{6}{7}+\frac{67}{126}=8$.

3. $101\frac{1}{6}+315\frac{2}{7}+\frac{19}{28}=417\frac{11}{84}$.

4. $4\frac{3}{4}+5\frac{2}{3}+1\frac{5}{9}+12\frac{1}{5}=24\frac{31}{180}$.

5. $\frac{1}{8}+\frac{3}{11}+8\frac{1}{3}+19\frac{29}{40}+62\frac{7}{9}=91\frac{463}{1980}$.

SUBTRACTION OF FRACTIONS.

67. When the given fractions have a common denominator, we subtract the numerator of the one from the numerator of the other, and place the remainder over the common denominator, thus $\frac{6}{7}-\frac{5}{7}=\frac{1}{7}$. But when they have different denominators, we first reduce them to a common denominator, (art. 63 or 64,) because (art. 27) we cannot subtract units of a different order from each other, after which we procede as before.

EXAMPLES.

1. $\frac{6}{7}-\frac{2}{3}=\frac{18}{21}-\frac{14}{21}=\dfrac{18-14}{21}=\frac{4}{21}$.

2. $\frac{5}{9}-\frac{1}{5}=\frac{25}{45}-\frac{9}{45}=\dfrac{25-9}{45}=\frac{16}{45}$.

3. $\frac{17}{19}-\frac{8}{9}=\frac{153}{171}-\frac{152}{171}=\dfrac{153-152}{171}=\frac{1}{171}$.

4. $\frac{5}{6}-\frac{3}{8}=\frac{11}{24}$; $\frac{7}{8}-\frac{4}{5}=\frac{3}{40}$, and $\frac{11}{12}-\frac{5}{9}=\frac{13}{36}$.

5. $\frac{16}{51}-\frac{3}{17}=\frac{7}{51}$; $\frac{5}{13}-\frac{8}{61}=\frac{201}{793}$, and $\frac{103}{140}-\frac{1}{2}=\frac{33}{140}$.

SUBTRACTION OF MIXED NUMBERS.

68. If the fractional parts have different denominators, we first reduce them to a common denominator; we then write the less mixed number under the greater, and having drawn a line underneath, we subtract the numerator of the lower fraction from that of the upper one, placing the remainder over the common denominator for the difference of

the fractions. Lastly, we find the difference of the whole numbers as usual.

But when the numerator of the lower fraction is the greater of the two, we subtract it from the sum of the terms of the upper fraction, and having placed the remainder over the common denominator, we carry one to the unit figure of the lower number, and procede as usual.

EXAMPLES.

1. If from $7\frac{1}{8}$ I would subtract $2\frac{3}{7}$, I procede as follows: I first find that $\frac{1}{8}$ and $\frac{3}{7}$ are equivalent to $\frac{7}{56}$ and $\frac{24}{56}$; I then place the numbers under each other, thus:

$$\begin{array}{r} 7\frac{7}{56} \\ 2\frac{24}{56} \\ \hline 4\frac{39}{56} \end{array}$$

and because $24>7$, I reduce a unit (art. 59) to fifty-sixths, and add it to the 7 fifty-sixths. Now as the denominator 56 shows how many parts we conceive in the unit, (art. 58,) I shall have just as many to add to the 7 as are expressed by its denominator; I therefore add the denominator itself to the 7 and subtract 24 from the sum; or, which is the same thing, I say $56-24=32$, and $32+7=39$, which I place over the common denominator. Lastly, having added a unit to the upper number, I also add a unit to the lower number, saying 1 and 2 is 3 and 3 from 7 leaves 4. If we add the same number to each of two given numbers, their difference (art. 28) is still the same: $4\frac{39}{56}$ is therefore the difference between $7\frac{1}{8}$ and $2\frac{3}{7}$.

We may also reduce the mixed numbers to improper fractions, and perform the subtraction as in the preceding article. Thus $7\frac{1}{8}=\frac{57}{8}$, and $2\frac{3}{7}=\frac{17}{7}$; therefore $7\frac{1}{8}-2\frac{3}{7}=\frac{57}{8}-\frac{17}{7}=\frac{399}{56}-\frac{136}{56}=\frac{399-136}{56}$ $=\frac{263}{56}=4\frac{39}{56}$, as before.

2. $101\frac{13}{15}-94\frac{11}{13}=\frac{1528}{15}-\frac{1233}{13}=\frac{a}{d}-\frac{b}{d}=\frac{a-b}{d}=\frac{c}{d}=$ $7\frac{4}{195}$.

Or thus, $\frac{13}{15}$ and $\frac{11}{13}$ are equivalent to $\frac{169}{195}$ and $\frac{165}{195}$, then from $101\frac{169}{195}$

we take $94\frac{165}{195}$

and have $7\frac{4}{195}$ for the remainder as before.

Here $\frac{13}{15}>\frac{11}{13}$; we therefore do not add a unit to either of the numbers, as in the preceding example.

3. To subtract $94\frac{165}{195}$ from 101, I place the numbers thus:

$$\begin{array}{r} 101 \\ 94\frac{165}{195} \\ \hline 6\frac{2}{13} \end{array}$$

and, as there is nothing above $\frac{165}{195}$, I subtract it from a unit, and carry a unit to the 4, as in the first example. Now in reducing a unit (art. 59) to a fraction whose denominator shall be 195, I shall also have 195 for the numerator; but this is the same as the denominator of the fraction $\frac{165}{195}$; I therefore subtract the numerator of this fraction from its own denominator, and placing the difference 30 over the denominator, I have $\frac{30}{195}$ or (art. 62) $\frac{2}{13}$ for the difference between $\frac{165}{195}$ and a unit: after which, carrying a unit to the 4 and subtracting, I have 6; wherefore $6\frac{2}{13}$ is the difference between the given numbers.

The same may be applied to all similar examples.

NOTE. The difference between a fraction and a unit is called the *complement* of that fraction, thus $\frac{2}{13}$ is the complement of $\frac{11}{13}$.

4. $15\frac{3}{4}-3\frac{7}{8}=11\frac{7}{8}$ and $15\frac{3}{4}-11\frac{7}{8}=3\frac{7}{8}$.

5. $9\frac{31}{40}-6\frac{7}{8}=2\frac{9}{10}$ and $9\frac{31}{40}-2\frac{9}{10}=6\frac{7}{8}$.

6. $28\frac{2}{3}-9\frac{11}{12}=18\frac{3}{4}$ and $28\frac{2}{3}-18\frac{3}{4}=9\frac{11}{12}$.

7. $65\frac{5}{17}-19\frac{11}{12}=45\frac{77}{204}$ and $9-4\frac{3}{7}=4\frac{4}{7}$.

8. $101-6\frac{2}{13}=94\frac{11}{13}$ and $15\frac{1}{7}-14\frac{16}{21}=\frac{8}{21}$.

9. $1043\frac{7}{19}-65\frac{5}{17}=978\frac{24}{323}$. Prove this and the succeding example both by addition and subtraction.

10. $3171\frac{1}{17}-980\frac{7}{51}=2190\frac{47}{51}$.

MULPLICATION OF FRACTIONS.

69. To find the product of several fractions, we multiply all the numerators together for a new numerator, and all the denominators together for a new denominator. The fraction thus formed is the product of the given fractions.

For example, if we have $\frac{3}{4}$ to multiply by $\frac{2}{3}$, we shall first multiply $\frac{3}{4}$ by 2; to effect which, we multiply the numerator 3 by 2, (art. 61,) and we have $\frac{6}{4}$ for the product.

But $\frac{2}{3}$ is the same (art. 58) as a third part of 2, and it is easy to perceive that, in multiplying by a third part of 2, the product should be just $\frac{1}{3}$ of the product that we have in multiplying by 2; the product $\frac{6}{4}$ is then just 3 times the true product; we must therefore divide $\frac{6}{4}$ by 3, to effect which, (art. 61) we multiply its denominator 4 by 3, which gives $\frac{6}{12}$ for the quotient. Wherefore $\frac{6}{12}$ or $\frac{1}{2}$ is the product of $\frac{3}{4}$ multiplied by $\frac{2}{3}$. The operation is indicated thus:

$$\frac{3\times 2}{4\times 3}=\frac{6}{12}=\frac{1}{2}.$$

Or, to multiply $\frac{3}{4}$ by $\frac{2}{3}$, we shall first multiply $\frac{3}{4}$ by 2, to do which (art. 61) we divide its denominator 4 by 2, and have $\frac{3}{2}$ for the product. Now this product, as has been shown above, is just 3 times the true product: we shall therefore divide $\frac{3}{2}$ by 3, to effect which (art. 61) we divide its numerator by 3, which gives $\frac{1}{2}$ for the product of $\frac{3}{4}$ multiplied by $\frac{2}{3}$, as before. This operation is expressed thus:

$$\frac{\overset{1}{\cancel{3}}\times \cancel{2}}{\underset{2}{\cancel{4}}\times \cancel{3}}=\frac{1}{2}.$$

The reasoning here applied to two fractions will evidently apply to another fraction and their product, and consequently to the multiplication of any number of fractions.

Let us observe that, in multiplying by a fraction, we not only perform a multiplication but also a division, and that the number by which we divide is always greater than that by which we multiply; hence the quantity which is multiplied by a fraction, instead of being increased, is always diminished.

Again, to multiply the fractions $\frac{1}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$ together, having seen above that the product of any number of fractions is found by multiplying their numerators together and their denominators together, the multiplication of the given fractions is indicated thus:

$$\frac{1\times\cancel{3}\times\cancel{4}\times\cancel{5}}{\cancel{3}\times\cancel{4}\times\cancel{5}\times 6}=\frac{1}{6}.$$

The lines drawn through the numbers 3, 4, 5, show that they cancel each other, that is to say, because the value of a fraction (art. 58) is not altered when both its terms are multiplied by the same number, the value of $\frac{1}{6}$ would not be altered if both its terms were multiplied by the product of $3\times4\times5$. We therefore omit the multiplication of these numbers, by which means we save ourselves the trouble not only of this multiplication, but also of reducing the final product to its lowest terms.

Hence we may always cancel a numerator and a denominator when they are alike, because this is the same as to divide the two terms of the fractional product each by the same number, which (art. 58) does not alter its value. But if all the numerators become cancelled, we shall have a unit for the numerator of the product, because any number divided by itself gives a unit. It is the same with the denominators.

It is easy to perceive that we may divide a numerator by a denominator, or a denominator by a nu-

merator; also that a numerator and a denominator may both be divided by the same number; because, in either case, the effect will be the same as to divide each term of the product by the same number. Thus, to multiply $\frac{9}{13}$ by $\frac{26}{27}$:

$$\frac{\cancel{9} \times \overset{2}{\cancel{26}}}{\cancel{13} \times \underset{3}{\cancel{27}}} = \frac{2}{3}$$

Having indicated the multiplication by the sign into, as usual, I find that the numerator 9 of the first fraction will divide the denominator 27 of the second; also that 13, the denominator of the first, will divide 26, the numerator of the second: I therefore divide, and have $\frac{2}{3}$ for the product.

Now if we multiply the numerators together and the denominators together, we have $\frac{234}{351}$ for the product; and, reducing this product to its lowest terms by dividing each term by the difference between the two, we have $\frac{2}{3}$ as before.

EXAMPLES.

1. $\frac{1}{2} \times \frac{5}{6} \times \frac{7}{8} = \frac{35}{96}$, and $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} = \frac{5}{8}$.
2. $\frac{3}{7} \times \frac{5}{9} \times \frac{21}{25} = \frac{1}{5}$, and $\frac{4}{9} \times \frac{14}{17} \times \frac{51}{56} = \frac{1}{3}$.
3. $\frac{2}{7} \times \frac{6}{11} \times \frac{11}{12} = \frac{1}{7}$, and $\frac{18}{19} \times \frac{2}{9} \times \frac{57}{58} = \frac{6}{29}$.
4. $\frac{3}{7} \times \frac{2}{5} \times \frac{8}{9} \times \frac{7}{8} \times \frac{15}{16} = \frac{1}{8}$, and $\frac{5}{13} \times \frac{16}{17} \times \frac{21}{23} = \frac{1680}{5083}$.
5. $\frac{2}{3} \times \frac{3}{7} \times \frac{5}{8} \times \frac{9}{10} \times \frac{11}{16} \times \frac{41}{44} = \frac{369}{3584}$.

MULTIPLICATION OF MIXED NUMBERS.

70. We have seen (art. 61) that to multiply a fraction by a whole number, or, which is the same thing, a whole number by a fraction, we multiply the whole number and the numerator of the fraction together, and divide the product by the denominator. Therefore, to multiply 53 by $5\frac{3}{8}$ we procede thus:

$53\times5=265$, and $53\times\frac{3}{8}=\frac{53\times3}{8}=\frac{159}{8}=19\frac{7}{8}$: then $265+19\frac{7}{8}=284\frac{7}{8}$, which is the product.

Or, reducing $5\frac{3}{8}$ to an improper fraction, thus: $53\times5\frac{3}{8}=\frac{53}{1}\times\frac{43}{8}=\frac{2279}{8}=284\frac{7}{8}$, as before.

Again, if we have $53\frac{3}{8}$ to multiply by 5, we procede thus: $53\times5=265$, and $\frac{3}{8}\times5=\frac{3\times5}{8}=\frac{15}{8}=1\frac{7}{8}$: then $265+1\frac{7}{8}=266\frac{7}{8}$, which is the product.

Or, reducing $53\frac{3}{8}$ to an improper fraction, thus: $53\frac{3}{8}\times5=\frac{427}{8}\times\frac{5}{1}=\frac{2135}{8}=266\frac{7}{8}$, as before.

Hence, if we have $15\frac{2}{3}$ to multiply by $5\frac{7}{8}$, we may procede as follows:

	$15\frac{2}{3}$
	$5\frac{7}{8}$
Product of 15×5	75
——— of $\frac{2}{3}\times5$	$3\frac{1}{3}$ or $\frac{8}{24}$
——— of $15\times\frac{7}{8}$	$13\frac{1}{8}$ or $\frac{3}{24}$
——— of $\frac{2}{3}\times\frac{7}{8}$	$\frac{14}{24}$
Total Product	$92\ \frac{1}{24}$

Or, we may reduce the given numbers to improper fractions, and multiply as in the preceding article, thus, $15\frac{2}{3}\times5\frac{7}{8}=\frac{47}{3}\times\frac{47}{8}=\frac{2209}{24}=92\frac{1}{24}$, as before. This method is generally the most simple.

NOTE. We can always give a whole number a unit for a denominator, and procede with it as with a fraction.

EXAMPLES FOR PRACTICE.

1. $24\times\frac{2}{3}=16$; $36\times\frac{3}{4}=27$; $42\times\frac{5}{6}=35$, and $42\times\frac{6}{7}=36$.

2. $\frac{9}{13}\times2\frac{8}{9}\times\frac{1}{2}=1$, and $\frac{5}{8}\times\frac{6}{7}\times14=7\frac{1}{2}$.

3. $\frac{2}{3}\times3\frac{7}{11}\times2\frac{3}{4}\times\frac{13}{14}=6\frac{4}{21}$, and $33\frac{1}{3}\times10\frac{4}{5}=360$.

4. $627\frac{11}{12}\times45\frac{9}{10}=28821\frac{3}{8}$, and $27\frac{3}{4}\times16=444$.

5. $94\frac{8}{21} \times 77\frac{17}{43} = 7304\frac{584}{903}$, and $69\frac{4}{19} \times 39\frac{3}{25} = 2707\frac{49}{95}$.

71. Fractions of fractions are called *compound fractions.* Thus $\frac{2}{3}$ of $\frac{3}{4}$; $\frac{1}{2}$ of $\frac{5}{6}$ of $\frac{7}{8}$, etc. are compound fractions.

We have seen (art. 69) that to multiply $\frac{3}{4}$ by $\frac{2}{3}$ is to take $\frac{1}{3}$ of twice $\frac{3}{4}$; it is therefore (art. 58) to take twice the third part of $\frac{3}{4}$, or $\frac{2}{3}$ of $\frac{3}{4}$. Therefore the expressions $\frac{2}{3}$ of $\frac{3}{4}$; $\frac{1}{2}$ of $\frac{5}{6}$ of $\frac{7}{8}$; $\frac{1}{8}$ of $\frac{2}{3}$ of 9, and all similar expressions, signify nothing else than the product of the quantities contained in them. Thus $\frac{1}{8}$ of $\frac{2}{3}$ of $9 = \frac{1}{8} \times \frac{2}{3} \times \frac{9}{1} = \frac{3}{4}$.

DIVISION OF FRACTIONS.

72. *To divide a fraction by a fraction, we invert the fraction which is the divisor, and multiply the fraction which is the dividend by the divisor thus inverted.*

For example, to divide $\frac{1}{2}$ by $\frac{2}{3}$ we invert the fraction $\frac{2}{3}$, which becomes $\frac{3}{2}$, after which we multiply $\frac{1}{2}$ by $\frac{3}{2}$, as has been taught, (art. 69,) and we have $\frac{3}{4}$ for the quotient of $\frac{1}{2}$ divided by $\frac{2}{3}$. The operation is indicated thus: $\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$.

We shall easily perceive the reason of this rule, as follows:

In multiplying the denominator of $\frac{1}{2}$ by 2, we have $\frac{1}{4}$, which (art. 6[illegible]) is the quotient of $\frac{1}{2}$ divided by 2: but the divisor $\frac{2}{3}$ is a third part of 2, (art. 58,) and it is evident that, in dividing by a third part of 2, the quotient should be just 3 times the quotient that we have in dividing by 2; we must therefore multiply $\frac{1}{4}$ by 3, which is done in multiplying the numerator 1 by 3, (art. 61.) Therefore $\frac{3}{4}$ is the quotient of $\frac{1}{2}$ divided by $\frac{2}{3}$. The same reasoning will apply to any two fractional numbers.

73. When either of the given numbers is a whole number, we write it in the form of an improper fraction, by giving it a unit for denominator. Also, when there are mixed numbers, we reduce them to

improper fractions. For example, to divide 4 by $\frac{5}{6}$, we write $\frac{4}{1}\div\frac{5}{6}=\frac{4}{1}\times\frac{6}{5}=\frac{24}{5}=4\frac{4}{5}$.

To divide 87 by $6\frac{3}{7}$, we procede thus: $87\div 6\frac{3}{7}=\frac{87}{1}\div\frac{45}{7}=\frac{87}{1}\times\frac{7}{45}=\frac{609}{45}=13\frac{24}{45}$.

Also, if we have $7\frac{2}{7}$ to divide by $3\frac{3}{4}$, we procede thus: $7\frac{2}{7}\div 3\frac{3}{4}=\frac{51}{7}\div\frac{15}{4}=\frac{51}{7}\times\frac{4}{15}=\frac{204}{105}=\frac{68}{35}=1\frac{33}{35}$.

74. When there are compound fractions, we reduce them to simple fractions, that is to say, we find the product of the quantities contained in them, which product, if an improper fracfion, we leave in the form of a fraction, after which we procede as before.

For example, if we have $\frac{3}{5}$ of $\frac{5}{8}$ to divide $\frac{2}{7}$ of $9\frac{1}{2}$, we procede thus: $\frac{3}{5}$ of $\frac{5}{8}=\frac{3\times 5}{5\times 8}=\frac{3}{8}$: then, as $9\frac{1}{2}=\frac{19}{2}$, $\frac{2}{7}$ of $9\frac{1}{2}=\frac{2}{7}$ of $\frac{19}{2}=\frac{2\times 19}{7\times 2}=\frac{19}{7}$; and lastly, $\frac{3}{8}\div\frac{19}{7}=\frac{3}{8}\times\frac{7}{19}=\frac{21}{152}$, the quotient sought.

Examples for Practice.

1. $\frac{7}{8}\div\frac{1}{7}=6\frac{1}{8}$; $\frac{9}{10}\div\frac{1}{40}=36$, and $100\div\frac{1}{50}=5000$.
2. $\frac{4}{9}\div 2=\frac{2}{9}$; $\frac{4}{9}\div\frac{2}{3}=\frac{2}{3}$, and $\frac{2}{3}\div\frac{2}{9}=3$.
3. $\frac{13}{17}\div\frac{7}{9}=\frac{117}{119}$, and $\frac{4}{11}\div\frac{1}{2}$ of $\frac{3}{4}=\frac{32}{33}$.
4. $\frac{1}{3}$ of $9\frac{1}{13}\div\frac{6}{7}$ of $\frac{5}{6}$ of $2=2\frac{26}{225}$.
5. $758\frac{11}{16}\div 83\frac{2}{19}=3\frac{21891}{25280}$.
6. $1435\frac{7}{8}\div 62=23\frac{79}{496}$.
7. $37296\div 3\frac{11}{12}=9522\frac{18}{47}$.
8. $1435\frac{7}{8}\div 23\frac{79}{496}=62$.

OF THE DIVISION OF A WHOLE NUMBER BY THE FACTORS OF ANOTHER WHOLE NUMBER.

75. When the divisor is a composite number, and is the product of two numbers, neither of which excedes twelve, we first divide by one of the factors of the divisor, and then divide the quotient by the

other, which method is generally more concise than that of dividing by the whole divisor at once.

For example, if we have 165369 to divide by 45, as 45 is the product of 9×5, we first divide by one of these factors, (no matter which,) and then by the other, as follows:

$$5)165369$$

$$9)33073\frac{4}{5}$$

$$3674\frac{7}{9}+\frac{4}{45}$$

Having divided by 5, the quotient is $33073\frac{4}{5}$.

Again, in dividing the integral part of this quotient, which is 33073, by 9, we have $3674\frac{7}{9}$; and in dividing the fractional part $\frac{4}{5}$ by 9, we have $\frac{4}{45}$ (art. 61.) We have therefore $3674\frac{7}{9}+\frac{4}{45}$ for the final quotient.

But to have only one fraction, we shall add the two fractions $\frac{7}{9}$ and $\frac{4}{45}$ together. Now the denominator of the fraction $\frac{7}{9}$ is the second divisor, and the denominator of the fraction $\frac{4}{45}$ is the product of the two divisors; consequently, as 9 is contained in 45 as often as is expressed by the first divisor, to reduce $\frac{7}{9}$ to a fraction, whose denominator shall be 45, we must multiply the numerator 7 by the first divisor 5, which gives $\frac{35}{45}$, after which, adding the two fractions $\frac{35}{45}$ and $\frac{4}{45}$ together, we have $\frac{39}{45}$ for the sum. The numerator 39 of this sum is composed of the last remainder 7 multiplied by the first divisor 5, plus the first remainder 4; the denominator 45 being the product of the two divisors or whole number divided by, and it is easy to see that the same will be the case whatever be the two remainders, or the numbers divided by. Hence we have the following general rule:

When there are two remainders, multiply the last remainder by the first divisor, and to the product add the first remainder; under this sum write the product of the two divisors, and reduce the fraction, if necessary, to its lowest terms.

The same operation performed according to this rule, will stand thus:

$$
\begin{array}{l}
5)\underline{165369} \\
9)\underline{33073} \;.\;.\; 4 \\
\quad \underline{3674} \;.\;.\; 7
\end{array}
\Bigg\} \frac{7\times5+4}{45}=\frac{39}{45}=\frac{13}{15}.
$$

$$3674 \quad \frac{13}{15} \text{ Quotient.}$$

It is evident that $3674\frac{13}{15}$ is the quotient of 165369 divided by 45, or the forty-fifth part of 165369, because in taking $\frac{1}{9}$ of $\frac{1}{5}$, we have $\frac{1}{45}$. (art. 71.)

When there is but one remainder, it is easy to perceive that if this belongs to the first division, it will take the product of the two divisors for a denominator; and that if it belongs to the second division, it will take the second divisor for a denominator.

Examples for Practice.

1. $237543\div25=9501\frac{18}{25}$.
2. $768539\div54=14232\frac{11}{54}$.
3. $5970509\div63=94769\frac{62}{63}$.
4. $1084624\div56=19368\frac{2}{7}$.
5. $21165897\div49=431957\frac{4}{49}$.
6. $99392745\div35=2839792\frac{5}{7}$.
7. Divide 169281735 separately by each of the numbers 24, 32, 36, 48, 81, and find the difference between the sum of the quotients and the dividend.

Answer, $146619403\frac{833}{864}$.

QUESTIONS ON SECTION 9.

1. How are fractions added when they have a common denominator?
2. Upon what principle is this rule established?
3. How do we add fractions which have not a common denominator?

4. Why must fractions have a common denominator before we can add them?

5. By what different methods are mixed numbers added?

6. When two fractions have a common denominator, how do we subtract the one from the other?

7. Upon what principle is this rule established?

8. How do we procede when the denominators are different?

9. Why must the fractions have a common denominator before we can subtract the one from the other?

10. What different methods are there of subtracting mixed numbers, and which do you prefer?

11. How do we multiply fractions together?

12. Why do we procede thus? Let this be explained by an example.

13. What is meant by cancelling?

14. Upon what general principle is this established?

15. What two operations are always implied in multiplying by a fraction?

16. Why is a quantity always diminished when multiplied by a fraction?

17. By what different methods are mixed numbers multiplied, which do you prefer, and why?

18. What are compound fractions, and how are they reduced to simple fractions?

19. How is one fraction divided by another? Let the nature of this operation be explained by an example.

20. What operations are always implied in dividing by a fraction?

21. Why is a quantity always increased when divided by a fraction?

22. How do we divide mixed numbers: also, when one of the given numbers is a whole number?

23. When we have divided a whole number by the two factors of another, and have two remain-

ders, how do we find the true remainder? Let the reason of this be explained by an example.

24. How do we procede when there is but one remainder?

25. How do we prove that this method of dividing will always give the true result?

Section 10.

DECIMAL FRACTIONS.

76. The term *decimal* is derived from the Latin word *decem*, which signifies *ten*, and is therefore applicable to that which is numbered by tens.

Hence the basis of the numeration of whole numbers is decimal, because we conceive that each of the units of which any figure towards the left hand is composed, contains ten units of the next order on the right of it.

Pursuing this idea, we consider the single unit as being composed of ten smaller units, which constitute another order on the right of the units.

The units of this order are called *tenths*, because each of them is $\frac{1}{10}$ of a unit, and are represented in the same manner as whole units. But that the units and the tenths may not be confounded with each other, a comma is placed on the right of the units. Thus, to express *forty-five units* and *three tenths*, we write 45,3, and read *forty-five* and *three tenths*.

Again, we consider the *tenth* as being composed of ten smaller units, which constitute another order on the right of the tenths. The units of this order are called *hundredths*, each of them being $\frac{1}{10}$ of $\frac{1}{10}$, or $\frac{1}{100}$ of a unit. Thus the number 45,34 signifies *forty-five units*, *three tenths*, and *four hundredths*. But as 3 tenths are equal to 30 hundredths, we read *forty-five* and *thirty-four hundredths*. The figures on the right of the comma are those which we call decimals, those on the left always retaining the name of whole numbers or integers.

Continuing thus to consider the unit of each order as containing ten units of the next order on the right hand, we conceive an infinity of different orders on the right of the comma. These orders, beginning with the tenths and proceding towards the right hand, are called *tenths*, *hundredths*, *thousandths*, *ten-thousandths*, *hundred-thousandths*, *millionths*, *ten-millionths*, *hundred-millionths*, *billionths*, etc.

77. The decimal figures on the right of the comma are always read in the same manner as whole numbers; but after reading them we pronounce the name of the decimal units of the last order. For example, to read the number 643,643, (always beginning with the units of the highest order,) we say, six hundred and forty-three, and six hundred and forty-three *thousandths*.

We shall easily perceive the reason of this rule by observing that the 6 which is in the place of tenths, is equal to 60 hundredths, because every unit in the place of tenths is equal to ten hundredths; or, it is equal to 600 thousandths, because every hundredth is equal to ten thousandths: for the same reason, the figure 4 in the place of hundredths is equal to 40 thousandths. Therefore, the number 643 on the right of the comma being composed of 600 thousandths plus 40 thousandths plus 3 thousandths, is 643 thousandths; and the same reasoning will apply to any number of decimals.

78. When there are no integers, a cipher is placed on the left of the comma. Thus, to express twenty-five hundredths, we write 0,25.

Also, when any of the intervening decimals is wanting, its place is supplied with a cipher. Thus, to express five hundredths, we write 0,05; to express five thousandths, we write 0,005, etc.

Because a figure standing in any place towards the left hand is ten times as great as if it stood one place farther to the right, we perceive that the value of each decimal figure is determined by its distance from the comma.

If we write 0,25, 0,05, and 0,005 as vulgar fractions, we have $\frac{25}{100}$, $\frac{5}{100}$, $\frac{5}{1000}$: hence we easily discover that the decimal figures, or those on the right of the comma, may always be considered as the numerator of a fraction, whose denominator is a unit followed by as many ciphers as there are decimal figures. Therefore, in order to change a decimal fraction to its equivalent vulgar fraction, we have only to give to the decimal figures the denominator signified by the concluding word in reading them, that is to say, a unit followed by as many ciphers as there are figures on the right of the comma; at the same time, omitting in the numerator the ciphers, if any, which are found between the comma and the first significant figure, because these only serve to determine the proper value of each figure, by showing how far it is removed towards the right hand.

Reciprocally, when a decimal fraction is given in the form of a vulgar fraction, to express it in its entire form, we must place the numerator on the right of a comma, so that there may be as many decimal figures as there are ciphers following the unit in the denominator. For example, to express $\frac{75}{10000}$ in its entire form, because there are four ciphers following the unit in the denominator, and only two figures in the numerator, two ciphers must be placed on the left of 75, that is to say, between 75 and the comma; we therefore write 0,0075.

What we have just said greatly facilitates the writing of decimals, for, as the concluding word in reading a decimal fraction always shows what its denominator would be if expressed as a vulgar fraction, we have only to write the number preceding this word as a whole number, and place the comma on the left, so that the number of decimal figures may equal the number of ciphers following the unit in the number signified by the concluding word. For example, to write ninety-five-*millionths*, I first write 95 as a whole number; then, as the concluding word shows that the denominator of the fraction would be

1000000, and as this number contains six ciphers, I write four ciphers on the left of 95, and prefixing the comma, I have 0,000095 for the required decimal.

79. The value of decimal figures is not altered when we place ciphers on the right of them. Thus 0,25 is equal to 0,250, or 0,2500, or 0,25000, etc. and this is evident, in the first place, because these expressions are equivalent to $\frac{25}{100}$, $\frac{250}{1000}$, $\frac{2500}{10000}$, $\frac{25000}{100000}$, etc. and because (art. 58,) a fraction is not altered when both its terms are multiplied by the same number; and lastly, because the value of each figure is (art. 78,) determined by its distance from the comma. Hence, ciphers which are found on the right of a decimal fraction may be omitted.

80. As the value of each figure, whether integral or decimal, is determined by the place in which it stands towards the right or left, that is to say, by its distance from the comma, it is evident that, in any number containing decimals, if we remove the comma one or more places towards the right or left, the value of the number will be greater or less accordingly.

For example, if we take the number 234,567, and if, in removing the comma one place towards the right hand, we write 2345,67, it is evident that by this remove, the hundreds have become thousands, the tens hundreds, the units tens, the tenths units, the hundredths tenths, and the thousandths hundredths; that is to say, the value of each figure is ten times its former value, and consequently the number is multiplied by ten. Therefore, in removing the comma two places towards the right, we shall multiply by a hundred; in removing it three places, by a thousand, etc.

Again, if we take the same number 234,567, and if, instead of removing the comma towards the right, we remove it one place towards the left, thus 23,4567, it is evident that by this remove the hundreds have become tens, the tens units, the units *tenths, etc.* and therefore as each figure is only one-

tenth of its former value, the number is only one-tenth of its former value. Hence we perceive, that to divide by 10, 100, 1000, etc. we have only to remove the comma one, two, three, etc. places towards the left.

Let the following numbers be expressed in words.

1. 0,235.
2. 0,7089.
3. 0,61789.
4. 7,08017.
5. 91,002009.
6. 310,1065203.
7. 70019,00700311.
8. 0,0034580275.
9. 30375,000020067.
10. 0,81302000625031171311132569.

Let the following numbers be expressed in figures. (See art. 78.)

1. Nine units and twenty-seven *thousandths*.

2. Six hundred and three units and ninety-five *ten-thousandths*.

3. Eighteen units and twelve thousand six hundred and four *millionths*.

4. Ten thousand and twenty-five *ten-millionths*.

5. Five millions sixty thousand and eleven *billionths*.

6. Forty millions six hundred units, and ninety-seven thousand three hundred and fifty-eight *hundred-millionths*.

7. Sixty-four billions, forty-two millions, eleven thousand and seventeen *hundred-billionths*.

8. Change nine units and twenty-seven *thousandths* to *millionths*, and say how much the number is lessened by this change.

9. Change five hundred and four units and sixty-seven *ten-thousandths* to *hundred-millionths*, and say how much the number is lessened by this change.

10. Write the number seven units, thirty-six thousand and twenty-five *millionths*, and say how much this number would be diminished by changing it to *hundred-billionths*.

ADDITION OF DECIMALS.

81. As the unit of any order, whether on the right or left of the comma, contains ten units of the next order on the right of it, we add numbers containing decimals, or decimals alone, in the same manner as whole numbers; that is to say, we add units of the same kind to each other, and for every ten in the sum of any order of units, we carry one to the next order on the left.

NOTE. In placing units of the same order under each other, the commas will always stand under each other.

For example, if we would add 54; 75,2; 95,56, and 0,273 together, I write the numbers under each other so that units may be under units, tenths under tenths, etc. thus:

```
54,0
75,2
95,56
 0,273
-------
225,033
```

I then procede as usual, but having added the tenths, I place the comma on the left of the figure which I write down; or, which is the same thing, under the other commas. Having added the whole, I have 225,033 for the sum. The work is proved by adding both ways.

EXAMPLES FOR PRACTICE.

1. $376,3+5,674+0,23+150=532,204$.
2. $0,36+0,536+789,3+1,16=791,356$.
3. $373+25,25+0,789+236,1+5,4=640,539$.
4. $0,234+0,5+0,567+0,462+0,0005=1,7635$.
5. $532,204+791,356+640,539+1,7635=1965,8625$.

The scholar may, if he pleases, prove this last

example by adding the several numbers which compose the four preceding examples.

SUBTRACTION OF DECIMALS.

82. To subtract numbers containing decimals, we place the less number under the greater, so that units of the same order may be under each other; after which, we subtract as in whole numbers, taking care to place the comma on the left of the tenths, or under the other commas, as in addition.

When one of the given numbers has more decimal figures than the other, we may render the number of decimals the same in each, by placing as many ciphers as are necessary on the right of that number which has the fewest decimals, which (art. 79,) will not alter its value.

For example, to subtract 78,358 from 80,3, I write the numbers according to the rule, thus:

$$\begin{array}{r} 80{,}300 \\ 78{,}358 \\ \hline 1{,}942 \end{array}$$

after which, I subtract as usual, and placing the comma on the left of the tenths, I have 1,942 for the remainder. Let this and the succeding examples be proved both by addition and subtraction.

Examples for Practice.

1. 652,34—58,603=593,737.
2. 1009,3—9,6389=999,6611.
3. 711,823—70,9=640,923.
4. 21000—0,5001=20999,4999.
5. 1—0,99951807324=0,00048192676.
6. What is the difference between the sum of the answers to the five preceding examples, and one hundred thousand?

Answer. Seventy-six thousand seven hundred and sixty-six units and seventeen billions, eight hundred

and fifty-one millions, eight hundred and seven thousand three hundred and twenty-four *hundred-billionths.*

MULTIPLICATION OF DECIMALS.

83. We perform the multiplication of numbers containing decimals as if they were whole numbers; but in the product we separate on the right hand by a comma, as many figures as there are decimals in both the factors.

For example, to multiply 7,54 by 6,2, I place the numbers without any regard to the commas, thus:

$$\begin{array}{r} 7{,}54 \\ 6{,}2 \\ \hline 1508 \\ 4524 \\ \hline 46{,}748 \end{array}$$

and having multiplied as usual, and proved the work by casting out the nines, I separate three figures for decimals in the product, because there are three decimal figures in the two factors.

Because 754 is 100 times the given multiplicand 7,54, we easily perceive that in taking 754 any number of times, the product will be 100 times the product which we should have in taking 7,54 the same number of times.

Again, because 62 is ten times the given multiplier, 6,2, in multiplying by 62 we have ten times the product which we should have in multiplying by 6,2.

The product 46748, which we find in multiplying the given numbers, as if they were whole numbers, is therefore 10 times 100 or 1000 times the true product; we therefore divide this product by 1000, in separating three figures on the right hand by a comma, (art. 80,) that is to say, we separate as many of the right hand figures of the product as there are decimal figures in both the factors.

When the number of figures in the product is not so great as the number of decimals in both factors, we place as many ciphers between the figures of the product and the comma as will complete the number required by the rule.

For example, to multiply 0,17 by 0,5:

$$\begin{array}{r} 0{,}17 \\ 0{,}5 \\ \hline 0{,}085 \end{array}$$

Having multiplied the number 17 by 5, we have 85 for the product. But if we apply the same reasoning as in the preceding example, we find that because 17 is 100 times the given multiplicand, and because 5 is ten times the given multiplier, the product 85 is 1000 times the true product. The true product is therefore 85 divided by 1000, which (art. 80) is the same as 85 thousandths; we therefore write 0,085.

By vulgar fractions $\frac{17}{100}\times\frac{5}{10}=\frac{85}{1000}$.

Examples for Practice.

1. 0,153×0,82×0,017=0,00213282.
2. 7,813×3,69×2,17=
3. 11,917×1261,4×2,84=
4. 0,05418×4,816×75,15=
5. 1926,48×173,4×1,632=

DIVISION OF DECIMALS.

84. We have seen, in the above article, that the product must have as many decimals as are contained in both factors. Now in division the dividend is the product of the divisor and quotient, (art. 45;) therefore, in the division of numbers containing decimal parts, the divisor and quotient together must contain as many decimals as are contained in the dividend.

When the figures of the dividend are insufficient to contain those of the divisor, place any number of

ciphers at pleasure as decimals on the right of the dividend, which (art. 79) will not alter its value; then divide as in whole numbers, and separate by a comma a number of the right hand figures of the quotient equal to the number by which the decimal figures of the dividend excede those of the divisor.

Should the number of figures in the quotient be less than the number by which the decimals of the dividend excede those of the divisor, supply the defect by placing a sufficient number of ciphers between the figures of the quotient and the comma.

For example, to divide 0,654 by 298,6:

```
298,6)0,654000(0,00219+
        5972
        ----
         5680
         2986
        -----
         26940
         26874
         -----
            66
```

As the figures of the dividend are not sufficient to contain the divisor, I place three ciphers on the right of the former, which ciphers (art. 79) have no effect upon their value. I then divide as usual, and have 219 for the quotient. Now because the dividend (which is the product of the divisor and quotient) contains six decimals, and the divisor but one, there must be five decimals in the quotient; but as this quotient contains only three figures, I place two ciphers between these and the comma: I have therefore 0,00219 for the quotient of 0,654 divided by 298,6. As there is a remainder, it is easy to perceive that this division might be carried farther by annexing more ciphers to the dividend. The quotient is therefore somewhat too small, for which reason the sign + is placed on the right of it, to show that something more might be added; but as the last *figure*, 9, signifies hundred thousandths, and as the

next figure, if the division were continued, would be a cipher, the quotient 0,00219 is not a millionth of a unit less than the true quotient.

The correctness of the operation may be proved either by multiplication or by casting out the nines.

Examples for Practice.

1. 0,0658÷3256=0,000020208+

Ciphers which are placed between decimal figures and the comma, serve to determine the value of these figures, by showing their position towards the right; but as we divide the different orders of decimals in the same manner as whole numbers, we take no notice of such ciphers, whether in the dividend or divisor, till we count the number of decimals in each, in order to determine the place of the comma in the quotient.

2. 0,654÷0,00219=298,6301+
3. 0,654÷0,002191=298,4938+
4. 637,5÷8,5=75, and 63,75÷0,85=75.
5. 23÷0,71875=32, and $\frac{61}{16}$=3,8125.
6. 149,172÷0,496=300,75.
7. 63261÷64,8=976,25.
8. 52095,10857÷6358,49=8,193.
9. 3,141593÷0,7854=3,9999+

85. When the number of decimals in the dividend and divisor is the same, we may suppress the comma in both, which will not in any respect alter the quotient; because if each of them contains one, two, three, etc. decimal figures, in suppressing the comma (art. 80) each of them will become 10, 100, 1000, etc. times its former value, and (art. 43) the quotient is not altered when the dividend and divisor are both multiplied by the same number.

Also, when one of the given numbers contains more decimals than the other, we can easily render the number of decimals the same in both, as in subtraction, (art. 82.)

For example, to divide 234,9 by 3,31, as the divi-

sor contains two decimals and the dividend but one, I place a cipher on the right of 9 tenths in the dividend, which (art. 79) does not alter its value; I have then 234,90 to divide by 3,31, or, suppressing the comma in both, 23490 to divide by 331.

$$
\begin{array}{r}
331)23490(70\frac{320}{331} \\
2317 \\
\hline
320
\end{array}
$$

Having performed the division, I have $70\frac{320}{331}$ for the quotient.

But as the object of decimals, wherever they are used, is generally to avoid the use of vulgar fractions, we shall continue the division, by annexing ciphers, as follows:

$$
\begin{array}{r}
331)23490(70,966+ \\
2317 \\
\hline
3200 \\
2979 \\
\hline
2210 \\
1986 \\
\hline
2240 \\
1986 \\
\hline
254
\end{array}
$$

Having found the integral part of the quotient, as before, I place a comma on the right of it, after which I reduce the remainder 320 to tenths (see remark 25) by placing a cipher on the right of it, and dividing these tenths by 331, I have 9 tenths for the quotient, which I place on the right of the comma. I then reduce the 221 tenths which remain to hundredths, by placing a cipher on the right, and dividing these hundredths by 331, the quotient is 6 hundredths, which I place on the right of the 9 tenths; and in this manner the division may be continued without end.

Having continued the division to three places of decimals, we have 70,966 for the quotient, that is to say, we have 0,966 instead of the fraction $\frac{320}{331}$. The quotient 0,966 is evidently less than the value of the fraction $\frac{320}{331}$, but as the last figure of this quotient signifies thousandths, it is not the thousandth part of a unit less. Now, though we cannot obtain in decimals the exact value of $\frac{320}{331}$, yet it is easy to perceive that by continuing the division we can find the value of this fraction within any assignable quantity.

Again, if we have 0,0065 to divide by 0,73, having rendered the number of decimals equal in these two numbers, and suppressed the comma, we have 0065 or 65 to divide by 7300.

```
7300)65,000(0,008904+
     58400
     -----
      66000
      65700
      -----
        30000
        29200
        -----
          800
```

But three ciphers must be placed on the right of the dividend before it will contain the divisor. Now it is evident that 65 units are equal to 650 tenths, 6500 hundredths, or 65000 thousandths, etc.; therefore, as we divide 65000 instead of 65, we must (art. 25) consider the dividend as being reduced to thousandths, and consequently the quotient figure 8, which we find in dividing these thousandths by 7300, signifies thousandths. Hence we easily determine the place of the comma, after which we continue the division at pleasure, as in the preceding example.

Because (art. 43) the quotient is not altered when the divisor and dividend are both divided by the same number, if we divide 650 by 73 we shall have the same quotient that we have in dividing 65000 by 7300; therefore, having first determined the place

of the comma, we may always omit any number of ciphers on the right of the divisor.

EXAMPLES FOR PRACTICE.

1. $194 \div 0{,}358 = 194000 \div 358 = 541\frac{161}{179}$.
2. $9{,}318 \div 8242 = 0{,}00113+$
3. $21{,}4 \div 0{,}536 = 39{,}92537+$
4. $0{,}2673 \div 39 = 0{,}0068538461 5+$

QUESTIONS ON SECTION 10.

1. Whence do we derive the term decimal, and to what is it applied?
2. Is the numeration of whole numbers decimal? How is the unit divided in order to form the parts called decimals?
3. How do we distinguish decimals from integers?
4. What are the names of the different decimal orders, and how are they arranged?
5. What do we place on the left of the comma when there are no decimals?
6. How do we write five thousandths?
7. How do we express decimal fractions in the form of vulgar fractions?
8. How do we write a decimal fraction in its proper form which is written as a vulgar fraction?
9. How do we procede in writing decimals generally?
10. Why do decimal figures retain the same value when ciphers are placed on the right of them?
11. How do we determine the value of any particular decimal figure?
12. If in a decimal number we advance the comma one, two, three, etc. places towards the right or left, what is the effect?
13. How do we place decimal numbers under each other in order to add them, and why thus?
14. When the numbers are so placed, how are the commas situated?

15. Why do we carry a unit to the next order for every ten which we find in the sum of any order of decimals, just as in whole numbers?

16. To subtract one decimal number from another, how do we place the numbers, and why?

17. How does it happen that we procede in the same manner in the subtraction of decimals as in the subtraction of whole numbers?

18. How do we multiply decimal numbers?

19. Do we regard the comma in multiplying?

20. How do we place the comma in the product, and why? Let this be explained by an example.

21. How do we place the comma when the number of figures in the product is not so great as the number of decimals in both factors?

22. How do we divide decimal numbers, supposing the divisor to be greater than the dividend?

23. What must be observed with regard to the comma?

24. Why must the divisor and quotient together contain as many decimals as are contained in the dividend?

25. How do we procede when there are ciphers between the comma and the significant figures of the divisor?

26. Why is the sign plus sometimes placed on the right of the quotient?

27. How can we suppress the comma in the divisor and dividend before we commence the operation?

28. How does it appear that by this method we shall always obtain the true quotient?

Section 11.

86. A vulgar fraction is reduced to a decimal fraction in dividing the numerator by the denominator, as above.

Now this is the same as to reduce the given vulgar fraction to another, whose denominator (art.

78) shall be one of the numbers 10, 100, 1000, etc. Wherefore, if the denominator of the given fraction be not an aliquot part of one of the numbers 10, 100, 1000, etc. we cannot obtain the exact value of this fraction in decimals, though we can (art. 85) approach as near to it as we please.

Examples.

1. To find the value in decimals of the fraction $\frac{5}{8}$, we divide the numerator by the denominator, thus:

$$8)5{,}000$$
$$\overline{0{,}625}$$

As there is no remainder, 0,625 is the exact value of $\frac{5}{8}$.

2. $\frac{1}{4}=0{,}25$; $\frac{1}{2}=0{,}5$; $\frac{3}{4}=0{,}75$, and $\frac{3}{8}=0{,}375$.

3. $\frac{63}{65}=0{,}96923076+$, and $\frac{1}{3}=0{,}3333$, etc.

87. When the value of a fraction cannot be exactly expressed in decimals, we find, by sufficiently extending the division, a periodical return of the same figures in the quotient. Thus, in finding the value of $\frac{7}{11}$, we have 0,6363, where in continuing the division the figures 6 and 3 will succede each other without end. The reason of this will appear evident, as follows: the remainder must be one of the numbers 1, 2, 3, etc. but cannot equal the divisor; therefore, if we have a different remainder at each division, we cannot perform as many divisions as there are units in the divisor without falling upon some remainder that we have had before, in which case the quotient figures will return in the same order.

88. If we subtract 9, 99, 999, etc. from 10, 100, 1000, etc. the remainder will always be a unit. Therefore, in dividing a unit by 9, 99, 999, etc. the only significant figure in the quotient will be a unit. Thus we see that $\frac{1}{9}=0{,}1111$; $\frac{1}{99}=0{,}0101$; $\frac{1}{999}=$ *0,001001*, etc.

Now we may consider the periodical decimal fraction 0,6363 as the product of 0,0101, etc. multiplied by 63. But 0,0101, etc. is equal to $\frac{1}{99}$; therefore 0,6363, etc. is equal to $\frac{1}{99}$ multiplied by 63, that is to say, to $\frac{63}{99}$.

In the same manner we shall find that any periodical decimal fraction is equivalent to the vulgar fraction of which the numerator is a single period, and the denominator as many nines as there are figures in the period. Thus, 0,568568, etc. is equivalent to $\frac{568}{999}$.

It frequently happens that the period does not commence with the first decimal figure, in which case we consider the figures on the left of the period as units, and writing the vulgar fraction equivalent to the period on the right of these units, we reduce the whole as a mixed number to its equivalent improper fraction. We then place as many ciphers on the right of the denominator of this improper fraction, as there are decimal figures between the period and the comma.

For example, if we would reduce 0,325656 to its equivalent vulgar fraction, as the period is equal to $\frac{56}{99}$, the fraction 0,325656 is equivalent to $0,32\frac{56}{99}$.

Then suppressing the comma, and reducing $32\frac{56}{99}$ to an improper fraction, we have $\frac{3224}{99}$. But in suppressing the comma, (art. 80,) we render the number 100 times its former value; we must therefore divide $\frac{3224}{99}$ by 100, which is done (art's. 38 and 61,) in placing two ciphers on the right of the denominator 99. We have therefore $\frac{3224}{9900}$ for the fraction equivalent to 0,325656, etc.

To distinguish the period with facility, if it contains only one figure, a point is placed over this figure; also, if it contains more figures than one, it is distinguished by two points, one of which is placed over the first figure, and the other over the last.

EXAMPLES FOR PRACTICE.

1. $6\dot{1}=\frac{55}{90}$, or $\frac{11}{18}$, and $0{,}\dot{8}\dot{1}=\frac{9}{11}$.
2. $0{,}\dot{8}4\dot{6}=\frac{846}{999}$, or $\frac{94}{111}$.
3. $0{,}23\dot{5}=\frac{53}{225}$, and $0{,}15\dot{3}\dot{7}=\frac{761}{4950}$.
4. $0{,}6\dot{3}25\dot{8}=\frac{3514}{5555}$.

89. Decimals enable us to find, by multiplication, in a very expeditious manner, the quotient of any number divided by 5, or by any of those numbers in which 5 is the only factor.

For example, to divide 394678 by 5, we find the quotient by multiplication as follows. In dividing a unit by 5, we have 0,2; and consequently in dividing 394678 by 5, we shall for every unit in the dividend have 0,2 in the quotient. The quotient is therefore $0{,}2\times 394678$, or 78935,6.

Again, if we would divide 394678 by 5×5, or 25, we have only to multiply by $0{,}2\times 0{,}2$, or 0,04.

For, in order to divide the given number by 25, we have only (art. 75,) to take $\frac{1}{5}$ of $\frac{1}{5}$ of it. But $\frac{1}{5}$ of $\frac{1}{5}$ is the same (art. 58,) as $\frac{2}{10}$ of $\frac{2}{10}$, and consequently (art. 71,) the same as $\frac{4}{100}$. Therefore, to divide 394678 by 25 is the same as to take $\frac{4}{100}$ of it, that is to say, (art. 71,) the same as to multiply it by $\frac{4}{100}$, or (art. 78,) by 0,04. Therefore, the quotient

$$\frac{394678}{25}=394678\times 0{,}04=15787{,}12.$$

Thus we may form the following table.

To divide	by 5	we multiply	by	0,2
——	25	——		0,04
——	125	——		0,008
——	625	——		0,0016
——	3125	——		0,00032
——	15625	——		0,000064
	etc.			etc.

Where the whole numbers are formed by the continual multiplication of 5, and the corresponding decimals by the continual multiplication of 0,2.

90. We have seen (art. 43,) that when the product of any two numbers is divided by one of them, the quotient is the other. But the product of any two corresponding numbers in the above table is a unit, wherefore, in dividing a unit by any of the decimal numbers, we shall have the corresponding whole number for the quotient.

Hence we discover, that to multiply a number by any of the numbers 5, 25, 125, etc. we have only to divide by its corresponding decimal.

For example, if we have 34792 to multiply by 125, we divide 34792 by 0,008, according to article 85, and we have $\frac{34792000}{8}=4349000$ for the required product.

91. A number is called *abstract* when it is not applied to any particular species of quantity: thus, when we say 3, or 3 times, the number 3 is abstract.

92. When the number is applied to a particular species, as when we say 3 books, 20 bushels, etc. it is then called a *concrete* number.

93. Hitherto we have treated of numbers only in an abstract manner; but we shall shortly consider their application to the measurement and valuation of quantities. We have already observed (art's. 2 and 3,) that to measure a quantity, we must compare it with some known quantity of the same kind, which is called a unit. Now as quantities differ in their nature and magnitude, the units or measures to which they are compared vary accordingly.

94. A number which is made up of units of different magnitudes, is called a *compound number*. Thus, 6 pounds 15 shillings and 6 pence is a compound number, because the pound differs from the shilling, and each of these differs from the penny.

The following tables will show what relation the different units by which we usually measure quantities, have to each other.

FEDERAL MONEY OF THE UNITED STATES OF AMERICA.

10 mills, *m.* make	1 cent, *c.*
10 cents,	1 dime, *d.*
10 dimes, or 100 cents,	1 dollar, \$.
10 dollars,	1 eagle, *E.*

NOTE.—Accounts are kept in dollars and cents.

BRITISH MONEY.

Accounts are kept in England, Ireland, Canada, and some parts of the West Indies, and formerly were in the United States, in pounds, shillings, pence, and farthings; but the value of these denominations is very different in those countries.

4 farthings, *qr.* make	1 penny, *d.*	from	*denarius.*
12 pence	1 shilling, *s.*	—	*solidus.*
20 shillings	1 pound, £.	—	*libra.*

The farthings are written thus:

$\frac{1}{4}$ one farthing.
$\frac{1}{2}$ two farthings, or halfpenny.
$\frac{3}{4}$ three farthings.

The words *denarius*, *solidus*, and *libra*, are Latin, and signify penny, shilling, and pound, respectively.

PENCE TABLE.

d.		*s.*	*d.*		*s.*	*d.*
12	-	1	20	-	1	8
24	-	2	30	-	2	6
36	-	3	40	-	3	4
48	-	4	50	-	4	2
60	-	5	60	-	5	0
72	-	6	70	-	5	10
84	-	7	80	-	6	8
96	-	8	90	-	7	6
108	-	9	100	-	8	4
120	-	10	110	-	9	2
132	-	11	120	-	10	0
144	-	12	130	-	10	10

TABLE OF SHILLINGS.

s.		£	s.	s.		£	s.
20	-	1	0	80	-	4	0
30	-	1	10	90	-	4	10
40	-	2	0	100	-	5	0
50	-	2	10	120	-	6	0
60	-	3	0	150	-	7	10
70	-	3	10	200	-	10	0

TROY WEIGHT.

24 grains, *gr.* make	1 pennyweight, *dwt.*
20 pennyweights	1 ounce, *oz.*
12 ounces	1 pound. ℔.

Gold, Silver, and Jewels, are weighed by this weight.

APOTHECARIES WEIGHT.

20 grains, *gr.* make	1 scruple, ℈.
3 scruples	1 drachm, ʒ.
8 drachms	1 ounce, ℥.
12 ounces	1 pound, ℔.

Apothecaries mix their medicines by this weight, but buy and sell by Avoirdupois.

AVOIRDUPOIS WEIGHT.

16 drachms, *dr.* make	1 ounce, *oz.*
16 ounces,	1 pound, ℔.
28 pounds	1 quarter, *qr.*
4 quarters, or 112 pounds,	1 hundred wt., *cwt.*
20 hundred weight	1 ton, *T.*

By this weight are weighed all things of a coarse and drossy nature, Groceries, Chandler's wares, and all metals, except Gold and Silver.

NOTE.

A Faggot of Steel, is	120 lb.
A Fother of Lead	19 *cwt.* 2 *qrs.*
A Stone of Iron, Shot, or Horseman's weight,	14 lb.
A Stone of Wire	$10\frac{1}{2}$ lb.
A Barrel of Beef or Pork	200 lb.
A Barrel of Flour	196 lb.

LONG MEASURE.

10 lines English, or 12 lines French, *l.* make	1 inch, *in.*
12 inches	1 foot, *ft.*
3 feet	1 yard, *yd.*
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	1 rod or pole, *p.*
40 poles	1 furlong, *fur.*
8 furlongs, or 1760 yards	1 mile, *M.*
3 miles	1 league, *L.*
60 Geographical, or $69\frac{1}{2}$ Statute miles, nearly	1 degree, *deg.* or °.

360 degrees, or 25000 miles, nearly the circumference of the Earth.

NOTE.

A Fathom is 6 feet, and is used for Nautical measurements.

A Hand is 4 inches, and is used to measure the height of horses.

The Surveyor's, or four pole chain, is used to measure distances.

7,92 inches make	1 link.
100 links, 22 yards, or 66 feet,	1 chain.
80 chains	1 mile.

Long Measure is used where length only is considered.

CLOTH MEASURE.

$2\frac{1}{4}$ inches, *in.* make	1 nail, *n.*
4 nails	1 qr. of a yard, *qr.*
4 quarters	1 yard, *yd.*
$2\frac{1}{2}$ quarters, or 10 nails,	1 ell Hamburgh, *E. H.*
3 quarters	1 ell Flemish, *E. F.*
5 quarters	1 ell English, *E. E.*
6 quarters	1 ell French, *E. Fr.*

The name of this measure bespeaks its use.

LAND, OR SQUARE MEASURE.

144 square inches make	1 square foot, *S. F.*
9 feet	1 yard, *yd.*
$30\frac{1}{4}$ yards	1 pole, *P.*
40 poles in length, and 1 in breadth.	1 rood, *R.*
4 roods, or 160 poles,	1 acre, *A.*
640 acres	1 mile, *m.*

The Surveyor's, or Gunter's chain of 100 links, is used to measure land:—25 links make 1 rod, pole, or perch; 10 square chains make 1 acre.

NOTE. This measure respects length and breadth.

SOLID, OR CUBIC MEASURE.

1728 solid, or cubic inches, make	1 cubic foot, *C. F.*
27 feet	1 yard, *Y.*
40 feet of round, or 50 feet of hewn timber	1 ton, *T.*
Firewood, 8 feet long, 4 broad, and 4 high, or 128 cubic feet	1 cord, *C.*

By Solid Measure are measured such things as have length, breadth, and depth.

LIQUID MEASURE.

2 pints, *pts.* make	1 quart, *qt.*
4 quarts	1 gallon, *gal.*
$31\frac{1}{2}$ gallons	1 barrel, *bbl.*
42 gallons	1 tierce, *tier.*
63 gallons	1 hogshead, *hhd.*
84 gallons	1 puncheon, *pun.*
2 hogsheads, or 126 gallons,	1 pipe or butt, *p. b.*
2 pipes, or 252 gallons,	1 tun, *T.*

Brandy, Spirits, Cider, Vinegar, Molasses, Oil, etc. are measured by this measure. Honey is sold by the pound, avoirdupois.

231 cubic inches make a gallon, and 10 gallons make an anker.

DRY MEASURE.

2 pints, *pts.* make	1 quart, *qt.*
8 quarts	1 peck, *p.*
4 pecks	1 bushel, *bush.*
36 bushels	1 chaldron of coal, *ch.*

The diameter of a Winchester bushel is $18\frac{1}{2}$ inches, and its depth 8 inches. This measure is applied to dry articles, such as Corn, Fruit, Seed, Roots. Salt, Sand, Oysters, Coal, etc.

TIME.

60 seconds, *sec.* make	1 minute, *m.*
60 minutes	1 hour, *h.*
24 hours	1 day, *d.*
7 days	1 week, *w.*
4 weeks	1 month, *mo.*
13 months, 1 day, and 6 hours, or 365 days 6 hours,	1 common, or Julian year, *Y.*

NOTE. The odd six hours are not reckoned till they amount to a day; a common year, therefore, consists of 365 days, and every fourth, or leap year, of 366.

The year is also divided into twelve calendar months, which, in number of days, are unequal, as follows:

1st	month	January hath	31	days.
2d	——	February	28	—
3d	——	March	31	—
4th	——	April	30	—
5th	——	May	31	—
6th	——	June	30	—
7th	——	July	31	—
8th	——	August	31	—
9th	——	September	30	—
10th	——	October	31	—
11th	——	November	30	—
12th	——	December	31	—

Or, Thirty days hath September,
April, June, and November;
The other months have thirty-one,
Except the second month alone,
Which has but twenty-eight in fine,
Till leap-year gives it twenty-nine.

When the year is exactly divisible by 4, it is then leap-year, in which the second month (February,) has 29 days.

CIRCULAR MOTION.

60	seconds (″) make	1 prime, or minute, ′.
60	minutes	1 degree, °.
30	degrees	1 sign, *s*.

12 signs, or 360 degrees, make the whole circle of the Zodiac.

A TABLE,

Showing the value of the American and Spanish Dollar in the currency of the different States and British Dominions.

		s.	d.	
United States.	In Maine,	6	0	make a dollar.
	— Massachusetts,			
	— New-Hampshire,			
	— Vermont,			
	— Rhode Island,			
	— Connecticut,			
	— Virginia,			
	— Kentucky,			
	— Tennessee,			
	— New-York,	8	0	
	— North Carolina,			
	— New-Jersey,	7	6	
	— Pennsylvania,			
	— Delaware,			
	— Maryland,			
	— South Carolina,	4	8	
	— Georgia,			
British.	— Canada,	5	0	
	— Nova-Scotia,			
	— English, sterling,	4	6	
	— Irish,	4	10½	

PRACTICE TABLES.

Aliquot parts of a Shilling.

d.		s.
6	-	$\frac{1}{2}$
4	-	$\frac{1}{3}$
3	-	$\frac{1}{4}$
2	-	$\frac{1}{6}$
$1\frac{1}{2}$	-	$\frac{1}{8}$
1	-	$\frac{1}{12}$

Aliquot parts of a Cwt.

lb		Cwt.
56	-	$\frac{1}{2}$
28	-	$\frac{1}{4}$
16	-	$\frac{1}{7}$
14	-	$\frac{1}{8}$
8	-	$\frac{1}{14}$
7	-	$\frac{1}{16}$

Aliquot parts of a Pound.

s.	d.		£	s.	d.		£
10	0	-	$\frac{1}{2}$	1	4	-	$\frac{1}{15}$
6	8	-	$\frac{1}{3}$	1	3	-	$\frac{1}{16}$
5	0	-	$\frac{1}{4}$	1	0	-	$\frac{1}{20}$
4	0	-	$\frac{1}{5}$	0	8	-	$\frac{1}{30}$
3	4	-	$\frac{1}{6}$	0	6	-	$\frac{1}{40}$
2	6	-	$\frac{1}{8}$	0	4	-	$\frac{1}{60}$
2	0	-	$\frac{1}{10}$	0	3	-	$\frac{1}{80}$
1	8	-	$\frac{1}{12}$	0	2	-	$\frac{1}{120}$

A Table of Coins which pass current in the United States of North America, with their Sterling and Federal value.

Names of Coins.	Standing Weights.		Sterling Money of Great Britain.			Vermont, Maine, New-Hampshire, Massachusetts, Rhode Island, Connecticut, Virginia.			New-York and North Carolina.			New-Jersey, Pennsylvania, Delaware, and Maryland.			South Carolina and Georgia.			Federal value. Dollars.	Cents.	Mills.
(Gold.)	dwt.	gr.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.	D.	cts.	m.
A Johannes,	18	0	3	12	0	4	16	0	6	8	0	6	0	0	4	0	0	16	00	0
A half Johannes,	9	0	1	16	6	2	8	0	3	4	0	3	0	0	2	0	0	8	00	0
A Doubloon,	16	21	3	6	0	4	8	0	5	16	0	5	12	6	3	10	0	14	93	0
A Moidore,	6	18	1	7	0	1	16	0	2	8	0	2	5	0	1	8	0	6	00	0
An English Guinea,	5	6	1	1	0	1	8	0	1	17	0	1	15	0	1	1	9	4	66	7
A French Guinea,	5	5	1	1	0	1	7	6	1	16	0	1	14	6	1	1	5	4	60	0
A Spanish Pistole,	4	6	0	16	6	1	2	0	1	9	0	1	8	0	0	18	0	3	77	3
A French Pistole,	4	4	0	16	0	1	2	0	1	8	0	1	7	6	0	17	6	3	66	7
(Silver.)																				
An Eng. or Fr. Crown,	18	0	0	5	0	0	6	8	0	8	9	0	8	3	0	5	0	1	10	0
The Dollar of Spain, Sweden, or Denmark,	17	6	0	4	6	0	6	0	0	8	0	0	7	6	0	4	8	1	00	0
An English Shilling,	3	18	0	1	0	0	1	4	0	1	9	0	1	8	0	1	1	0	22	2
A Pistareen,	3	11	0	0	$10\frac{3}{4}$	0	1	2	0	1	7	0	1	6	0	0	11	0	20	0

All other Gold Coins, of equal fineness, at 89 cents per dwt., and Silver at 111 cents per oz.

A TABLE OF DIFFERENT FOREIGN COINS.

France.

12 Deniers	1 Sol,
20 Sols	1 Livre,
3 Livres	1 Crown.

Portugal.

400 Reas	1 Crusadoe,
1000 Reas	1 Mill-rea.

Italy.

12 Deniers	1 Sol,
10 Sols	1 Livre,
5 Livres	1 piece of eight at Genoa,
6 Livres	1 do. of Leghorn,
6 Solidi	1 Gross,
24 Grosses	1 Ducat.

Holland.

8 Pennings	1 Groat,
2 Groats	1 Stiver,
6 Stivers	1 Shilling,
20 Stivers	1 Florin or Guilder,
$2\frac{1}{2}$ Florins	1 Rix Doll.
6 Florins	1 £ Flemish,
5 Guilders	1 Ducat.

Denmark.

16 Shillings	1 Mark,
6 Marks	1 Rix Dollar,
32 Rustics	1 Copper Dollar,
6 Copper Dollars	1 Rix Dollar.

Russia.

18 Pennings	1 Gros,
30 Gros	1 Florin,
3 Florins	1 Rix Dollar,
2 Rix Dollars	1 Gold Ducat.

Spain.

4 Marvadies Vellon, or $2\frac{1}{8}$ Marvadies of Plate,	1 Quarte,
$8\frac{1}{2}$ Quartas, or 34 Marvadies Vellon,	1 Rial Vellon,
16 Quartas, or 34 Marvadies of Plate,	1 Rial of Plate,
8 Rial of Plate,	1 Piastre,
10 Rial of Plate,	1 Dollar,
5 Piastres,	1 Spanish Pistole.

The value of some Foreign Coins, etc. in Federal Money, as established by a late Act of Congress.

	Dls.	*Cts.*	*M.*
Pound Sterling,	4	44	4
Pound of Ireland,	4	10	0
Pagoda of India,	1	94	0
Tale of China,	1	48	0
Mill-rea of Portugal,	1	24	0
Ruble of Russia,	0	66	0
Rupee of Bengal,	0	55	5
Guilder of Holland,	0	39	0
Mark Banco of Hamburgh,	0	33	5
Livre of France,	0	18	5
Rial Plate of Spain,	0	10	0

95. The calculation of compound numbers often requires the conversion of units of a certain kind to units of a different kind, which operation is called *reduction.*

96. We convert units of a higher, to units of a lower name, by multiplication.

For example, to reduce £5 16*s.* 4*d.* to pence, as the pound is worth 20*s.* we multiply 5 by 20, which gives 100*s.* for the value of £5 in shillings: to this adding the 16*s.* we have 116*s.* for the value of £5 16*s.*

Again, because each shilling is worth 12 pence, we multiply 116*s.* by 12, which gives 1392 for the value of £5 16*s.* in pence, to which adding the 4 pence, we have 1396 for the value of £5 16*s.* 4*d.* in pence, as was required.

The operation performed at length is as follows:

```
 £  s.  d.
 5  16  4
20
———
116 value in shillings of £5 16s.
 12
———
1396 value in pence of £5 16s. 4d.
```

I have here added the 16s. in multiplying by 20: I have also added the 4 pence in multiplying by 12.

In the same manner, always beginning with the principal units, we can easily perform all similar reductions.

Again, to reduce 4*T.* 13*cwt.* 3*qrs.* 14lb to pounds, the operation is as follows:

```
T.   cwt.  qrs.  lb
4    13    3     14
20
--
93 value in cwts. of 4T. 13cwt.
 4
---
375 value in qrs. of 4T. 13cwt. 3qrs.
 28
----
3014
750
-----
10514 value in lbs. of 4T. 13cwt. 3qrs. 14lb,
```

as was required.

Examples for Practice.

1. In 5*s.* 3¾*d.* how many farthings?

Answer, 255.

2. In 76lb Troy how many grains?

Answer, 437760.

3. In 1 ton how many drachms?

Answer, 573440.

4. How many barley corns, three of which make one inch, will reach round the earth, supposing the circumference to be 25000 miles?

Answer, 4752000000.

5. How many seconds are there in a solar year, allowing it to be 365*d.* 5*h.* 48*m.* 57*sec.*?

Answer, 31556937.

97. Units of an inferior kind are converted to *those of* a superior kind by division.

For example, to have the value of 1396 pence, in pounds, shillings, and pence, as 12 pence make a shilling, it is evident that as often as 12 is contained in 1396, there will be so many shillings; we therefore divide by 12, and we have 116*s.* 4*d.* for the quotient.

Again, to bring 116*s.* to pounds, we divide by 20, because 20 shillings make £1, and we have £5 16*s.* for the quotient, which together with the 4*d.* gives £5 16*s.* 4*d.* for the value of 1396 pence, as was required.

The operation stands thus:

$$\begin{array}{r} 12)1396 \\ \hline 2,0)11,6s.\ \ 4d. \\ \hline £5\ 16s.\ 4d. \end{array}$$

To divide by 20 we divide (art. 75) first by 10 and then by 2. Now we divide a number by ten (art. 80) in separating the right hand figure by a comma; the figure separated is therefore tenths; but in dividing these tenths by 2 they will (art. 61) become twentieths, that is to say, twentieths of a pound, or shillings; we therefore consider the figure separated as signifying a number of shillings. Again, in dividing the figures on the left of the comma by 2, if there is a remainder, this remainder will always be 1, that is to say, ½ of a pound, or 10 shillings; we therefore write this remainder 1 on the left of the figure separated. In reducing cwts. to tons we divide by 20 in the same manner.

Hence we see that to reduce the units of any name to those of a superior name, we have only to divide by the number which shows how many of the smaller units make one of the next greater.

Examples for Practice.

1. In 10514 lbs. how many tons?

Ans. 4T. 13cwt. 3qrs. 14lb.

2. In 255 farthings how many shillings?

Ans. 5s. $3\frac{3}{4}d$.

3. In 437760 grs. Troy how many lbs.?

Ans. 76.

4. In 573440 drachms how many tons?

Ans. 1.

5. In 4752000000 barley corns how many miles?

Ans. 25000.

6. In 31556937 seconds how many days?

Ans. 365*d*. 5*h*. 48*m*. 57*sec*.

The above examples prove those of the preceding article.

98. A compound number, when reduced to its lowest denomination, becomes simple, its units being of the same kind. In this state it is susceptible of all the fundamental operations, in the same manner as an abstract number.

For example, to multiply 3*cwt*. 3*qrs*. 16lb by 343, in reducing 3*cwt*. 3*qrs*. 16lb to pounds (art. 96) we have 436lb.

Now the units of the number 436 being homogeneal, (that is to say, of the same kind,) it is evident that this number may be operated upon in the same manner as an abstract number; we therefore multiply 436 by 343, and we have 149548 for the product of the given numbers in pounds.

Also, if we reduce 149548lb to the higher denominations, as in the preceding article, we shall have 66*T*. 15*cwt*. 1*qr*. Therefore 3*cwt*. 3*qrs*. 16lb multiplied by 343 is equal to 66*T*. 15*cwt*. 1*qr*.

99. The unit of any of the lower denominations of a compound number is a fraction with regard to the principal unit, and the unit of each denomination is a fraction with regard to that of any of the superior denominations. Thus the shilling is $\frac{1}{20}$ of a pound; the penny is $\frac{1}{12}$ of the shilling, and, consequently, $\frac{1}{12}$ of $\frac{1}{20}$ or $\frac{1}{240}$ of a pound. Therefore any number of pence signifies so many twelfths when compared to a shilling, or so many two hundred and fortieths when compared to a pound.

Again, 1 oz. avoirdupois is $\frac{1}{16}$ of a lb, and 1 lb is $\frac{1}{28}$ of a qr.; therefore 1 oz. is $\frac{1}{16}$ of $\frac{1}{28}$ of a qr., but 1 qr. is $\frac{1}{4}$ of a cwt.; therefore 1 oz. is $\frac{1}{16}$ of $\frac{1}{28}$ of $\frac{1}{4}$ of a cwt.; that is to say, 1 oz. reduced to the fraction of a cwt. is equal to $\frac{1}{1}\times\frac{1}{16}\times\frac{1}{28}\times\frac{1}{4}=\frac{1}{1792}cwt.$

Hence we infer that to reduce the units of any denomination to the fraction of a higher denomination, we have only to place under these units as a denominator the number which shows how many of the smaller units make one of the greater. Thus, to reduce 440 yards to the fraction of a mile, as the mile is 1760 yards, we place this number under 440, and we have $\frac{440}{1760}$ or $\frac{1}{4}$ for the required fraction.

In effect, this is nothing else than to divide 440 yards or $\frac{440}{1}$ yards by 1760, (art. 61;) but, in dividing yards by 1760, we reduce them to miles, (art. 97:) the fraction $\frac{440}{1760}$ or $\frac{1}{4}$ is therefore the fraction of a mile.

100. From what has been said, we can easily reduce all the inferior parts of a compound number to a fraction of the principal unit. For example, to reduce 17*s*. 6*d*. to the fraction of a pound, we first reduce 6*d*. to the fraction of a shilling, in placing 12 underneath as a denominator, (art. 99,) which gives $\frac{6}{12}$ or $\frac{1}{2}s.$; we then have $17\frac{1}{2}s.$ or $\frac{35}{2}s.$ Now (art. 99) to reduce $\frac{35}{2}s.$ to the fraction of a pound, we divide by 20, that is to say, (art. 61,) we multiply the denominator by 20, which gives $\frac{35}{40}$ or $\frac{7}{8}$ for the fraction required.

Or thus: we reduce 17*s*. 6*d*. to pence, which gives 210*d*., under which placing 240, the number of pence in a pound, (art. 99,) we have $\frac{210}{240}l.=\frac{21}{24}$ $=\frac{7}{8}l.$ as before.

Now suppose that we would multiply 17*s*. 6*d*. by 315: instead of reducing 17*s*. 6*d*. to the lowest denomination, as in article 98, we reduce it to a fraction of the principal unit, that is to say, to the fraction of a pound; and multiplying this fraction by the given number 315, we have the answer in pounds at once.

Thus, 17*s*. 6*d*.$\times 315 = \frac{7}{8}l. \times \frac{315}{1} = \frac{2205}{8}l.$=£275 12*s*. 6*d*.

The division of £2205 by 8 is performed thus:

$$\begin{array}{r} 8)2205 \\ \hline £275\ 12s.\ 6d. \end{array}$$

Having divided £2205 by 8, we have £275 for the quotient, and a remainder of £5: this remainder we reduce to shillings, which gives 100*s*.

We then divide 100*s*. by 8, which gives 12*s*. for the quotient, and a remainder of 4*s*.: this remainder reduced to pence is 48*d*. and dividing 48*d*. by 8, we have 6*d*. for the quotient without remainder.

The quotient of £2205 divided by 8 is therefore £275 12*s* 6*d*.

101. From what has been said, it is easy to find the value of a fraction of a higher denomination in the terms of the inferior denominations. For example, to find the value of £$\frac{41}{96}$, as $\frac{41}{96}$ of a pound is the same as the ninety-sixth part of £41, (art. 60,) I divide £41 by 96, in the following manner:

$$\begin{array}{l} \quad\ £ \\ \quad\ 41 \\ \quad\ \ 20 \\ \hline 96)820(8s.\ 6\tfrac{1}{2}d. \\ \quad\ 768 \\ \hline \quad\ \ 52 \\ \quad\ \ 12 \\ \hline 96)624(6d. \\ \quad\ 576 \\ \hline \quad\ \ 48 \\ \quad\ \ \ 4 \\ \hline 96)192(2qrs.\ \text{or}\ \tfrac{1}{2}d. \\ \quad\ 192 \\ \hline \quad\ *** \end{array}$$

As £41 does not contain the divisor 96, we reduce £41 to shillings, after which, proceding as has been explained in the preceding article, we find that the value of £$\frac{41}{96}$ is 8*s.* 6$\frac{1}{2}$*d.*

EXAMPLES FOR PRACTICE.

1. Reduce 16*s.* 8*d.* to the fraction of a pound.
Ans. $\frac{5}{6}$.
2. What part of a month is 2*w.* 2*d.* 15*h.*?
Ans. $\frac{1}{2}\frac{3}{2}\frac{3}{4}$.
3. What part of a mile is 5*fur.* 13*p.* 1*yd.* 2*ft.* 6*in.*?
Ans. $\frac{2}{3}$.
4. What part of a cwt. is $\frac{2}{3}$ of an ounce avoirdupois?
Ans. $\frac{1}{2688}$.
5. What is the value of $\frac{3}{8}$ of a day?
Ans. 9 hours?
6. What is the value of $\frac{2}{3}$ of a mile?
Ans. 5*fur.* 13*p.* 1*yd.* 2*ft.* 6*in.*
7. What is the value of $\frac{2}{7}$ of a ton?
Ans. 5*cwt.* 2*qrs.* 24℔.

102. We can with equal facility reduce the lower denominations of a compound number to the decimal fraction of the principal unit, and this again to the terms of the lower denominations.

This is done in the same manner as in the preceding articles, except that we divide or multiply decimally. For example, to reduce 9*d.* to the decimal fraction of a shilling, we reduce 9*d.* to the vulgar fraction of a shilling, which gives $\frac{9}{12}$ or $\frac{3}{4}$; after which, reducing $\frac{3}{4}$ to a decimal, we have 0,75 for the fraction required.

Or, we divide 9 immediately by 12, thus:

$$\begin{array}{r} 12)9,00 \\ \hline 0,75 \end{array}$$

And in the same manner units of any denomination

are reduced to the decimal of a higher denomination.

Now to find the value in pence of 0,75 of a shilling, we multiply by 12, thus:

$$\begin{array}{r} 0{,}75 \\ 12 \\ \hline 9{,}00 \end{array}$$

and we have 9*d*. The one operation, therefore, proves the other.

Again, to reduce 5*s*. 6½*d*. to the decimal of a pound, the operation is as follows:

$$\begin{array}{l} 2)1{,}0 \\ \hline 12)6{,}5000d. \\ \hline 20)5{,}541\dot{6}s. \\ \hline £0{,}27708\dot{3} \end{array}$$

Having found 0,5 the value of ½*d*. we prefix the 6*d*. and we have 6,5*d*. for the value of 6½*d*: then, to have the value of 6,5*d*. in the decimal of a shilling, we divide by 12, and we have 0,541$\dot{6}$*s*., to which prefixing the 5*s*. we have 5,541$\dot{6}$*s*. for the value of 5*s*. 6*d*. Lastly, to reduce 5,541$\dot{6}$*s*. to the decimal of a pound, we divide by 20, as in article 97, that is to say, we remove the comma, or suppose it to be removed, one place towards the left, which (art. 80) is the same as to divide by 10, and dividing 0,5541$\dot{6}$ by 2, we have 0,27708$\dot{3}$ of a pound for the value of 5*s*. 6½*d*. as was required.

To find the value of 0,27708$\dot{3}$ of a pound in the terms of the lower denominations, we procede as follows:

$$
\begin{array}{r}
2,7708\dot{3} \\
2 \\
\hline
5,5416\dot{6} \\
12 \\
\hline
6,50000 \\
4 \\
\hline
2,0
\end{array}
$$

To find the value of $0,27708\dot{3}$ in shillings, we remove the comma one place towards the right, and multiply by 2, which (art. 80) is the same as to multiply by 10, and again by 2, that is to say, by 20; after which, as there are five decimals in the number $2,7708\dot{3}$, we separate as many in the product, (art. 83,) and we have 5,54166, etc. which is the same as 5*s*. and 0,54166, etc. of a shilling. Then, to find the value of 0,54166, etc. of a shilling, in pence, we multiply by 12: now, as the figure $\dot{6}$ taken any where in the expression 0,666, etc. is (art. 88) six ninths with regard to a unit of the next order on the left, we consider 72, the product of 12 times 6, as $\frac{72}{9}$ or 8; consequently, having multiplied the figure 6 of the number $0,5416\dot{6}$ by 12, we carry 8 to the next product, and, proceding as before, we have 6*d*. and 0,5 of a penny for the value of 0,54166, etc. of a shilling. Lastly, we multiply 0,5 of a penny by 4, and we have 2 farthings for the product. Wherefore, collecting the different denominations which we have thus found, we find that the value of $0,27708\dot{3}$ of a pound is 5*s*. $6\frac{1}{2}d$. This last operation, therefore, proves the preceding one.

By a similar process we reduce the inferior terms of a compound number of any other species, to the decimal of a higher denomination, and this decimal again to the inferior terms.

Examples for Practice.

1. Reduce 7*s*. 9¾*d*. to the decimal of a pound.
Ans. 0,390625.

2. What is the value of 0,375 of a month?
Ans. 1*w*. 3*d*. 12*h*.

3. Reduce 3*qrs*. 1*n*. to the decimal of an ell English.
Ans. 0,65.

4. Reduce 0,65 of an ell English to the decimal of a yard.
Ans. 0,8125.

5. What is the value of $0,\dot{2}8571\dot{4}$ of a ton?
Ans. 5*cwt*. 2*qrs*. 24℔.

103. The unit of each of the different denominations of federal money, from the highest to the lowest, being formed in the same manner as the unit of any order in an abstract number, that is to say, of ten units of the next inferior denomination, it is evident that the four fundamental operations may be performed upon these, as if they were abstract numbers.

To find the sum of $29,37 and $156,257, we place the units of the same denomination under each other, and add them as abstract numbers, thus:

$$\begin{array}{r} 29,37 \\ 156,257 \\ \hline 185,627 \end{array}$$

The lowest denomination being mills, we may (art, 77,) consider the sum as representing a number of mills.

Now, in dividing by the number of mills in a unit of any of the other denominations, it is evident, that we may read the sum in any or all of the other denominations; but, as accounts are kept in dollars and cents, it is usual to consider the dollars as whole numbers and the inferior denominations as decimals; therefore, as 1000 mills make a dollar, we divide by 1000 in separating (art. 80) three figures on the right *by a comma*, and read 185 dollars and 627 thou-

sandths, or, (as 10 mills make one cent,) 185 dollars, 62 cents, 7 mills.

Hence we procede in the calculation of federal money in the same manner as in the calculation of decimals.

EXAMPLES.

1. To find the sum of $5,16 ; $29,457 ; $347,20, and $1,627, we procede thus :

5,16
29,457
347,20
1,627
Sum $383,444

2. To subtract $597,237 from $1000,50, we procede thus:

$1000,50
597,237
Rem. $403,263

3. Multiply $35,637 by $1,25.
Ans. $44,54625.

4. Multiply $379,23 by $25. Ans. $9480,75.

5. Divide $35,84 by $6,25. Ans. $5,7344.

QUESTIONS ON SECTION 11.

1. How is a vulgar fraction reduced to a decimal?

2. Can we always express the exact value of a vulgar fraction in decimals?

3. Show when this can be done, when it cannot be done, and why.

4. When the value of a fraction cannot be exactly expressed in decimals, what do we find remarkable in extending the quotient?

5. What is the reason of this periodical return of the same figures?

6. How do we express the value of a periodical

decimal fraction in the terms of a vulgar fraction? Let this be explained by an example.

7. How do we procede when the period does not commence with the first decimal figure?

8. How is a period designated?

9. What expeditious method have you of multiplying or dividing a number by 5, or by any number in which 5 is the only factor?

10. What is an abstract number, and what a concrete number?

11. What is a compound number?

12. Repeat some of the tables which contain the different units to which we generally compare quantities, such as the tables of weight, measure, time, etc.

13. How is the dollar valued in the different currencies of the United States and British dominions?

14. By what operation do we convert units of a higher to units of a lower order?

15. How do we procede in order to reduce a compound number to its lowest denomination?

16. By what operation are units of an inferior order converted to those of a superior?

17. When a compound number is expressed in its lowest denomination, how do we express it in terms of the higher denominations?

18. What is a unit of any of the lower denominations, when compared to a unit of any of the higher?

19. How do we reduce all the inferior denominations of a compound number to a fraction of the principal unit, and how is this fraction again reduced to the terms of the inferior denominations? Let this be explained by an example.

20. How are the lower denominations of a compound number reduced to a decimal fraction of the principal unit?

21. How is this decimal fraction again reduced to the terms of the lower denominations?

22. What is there remarkable in the several divi-

sions of federal money? Let the scholar explain the nature of the calculation of this coin by examples chosen by himself.

Section 12.

ADDITION OF COMPOUND NUMBERS.

104. To perform this operation, we write the given numbers under each other, so that units of the same kind may be in the same column, and having drawn a line underneath, we commence by adding the units of the lowest denomination; if their sum does not contain a sufficient number of units to compose a unit of the next higher denomination, we write it under the units of its kind; but if it contains as many units as will compose one or more units of the next higher denomination, we reduce it (art. 97,) to this denomination, writing the remainder, if any, under the units of its kind, and adding the units of the quotient to their equals of the next higher denomination, with which we procede in like manner.

Example 1.

If we require the sum of

£	s.	d.
25	16	8
332	19	11
64	11	9
58	9	6
481	17	10

The sum of the pence is 34, which being reduced to shillings, (art. 97,) gives 2*s.* 10*d.*; I write the 10*d.* under pence, and add the 2*s.* to the unit column of shillings; the sum of this is 27, of which I set down the unit figure 7, and add the 2 to the column of tens; the sum of this is 5, and as each unit in this 5 is 10*s.*, it is evident that two of these will make a

pound; I therefore take the half of 5 for pounds, which gives 2 and 1 over; I place this 1, which is 10*s*. on the left of 7, and carry the 2 pounds to the column of pounds, which I add as usual.

Example 2.

We propose to add

	4	40	$30\frac{1}{4}$	9
A.	*R.*	*P.*	*Yd.*	*S. F.*
3	3	36	26	8
36	2	27	19	7
46	3	13	25	5
16	3	35	21	6
25	1	12	12	4
129	3	6	15	$5\frac{1}{4}$

Here the sum of the square feet is 30, which being reduced to square yards, (art. 97,) gives 3 *yds.* 3 *ft.*; I write the 3 *ft.* under the column of feet, and add 3 *yds.* to the unit column of yards; the sum of the yards is 106, and as $30\frac{1}{4}$ *yds.* make a pole, I divide 106 by $30\frac{1}{4}$, or (art. 60,) by $\frac{121}{4}$, that is to say, (art. 72,) I multiply 106 by $\frac{4}{121}$, which gives me $\frac{424}{121}$, or 3 poles and $\frac{61}{121}$ of a pole. Now reducing this fraction of a pole to yards, we have $\frac{61}{121}\times\frac{121}{4}=\frac{61}{4}=15\frac{1}{4}$ *yds.*; the sum total then of the yards is equal to 3 *P.* $15\frac{1}{4}$ *yds.*; I have therefore 15 *yds.* to write under yards, and 3 *P.* to carry to the column of poles. But before I procede to the addition of this column I shall take notice of the $\frac{1}{4}$ of a yard connected with the 15 *yds.* As it would be awkward to leave a fraction connected with the yards while we have a lower denomination in the question; I reduce this fraction to feet in multiplying it by 9, which gives $2\frac{1}{4}$ *ft.*, and adding this to the 3 *ft.* which I first placed under the column of feet, I have $5\frac{1}{4}$, which I leave under that column. I then procede to the addition of the poles, and adding the 3 *P.* which I retain from the reduction of the yards, I find that the sum

is 126; reducing this to roods, I have 3 *R.* 6 *P.*; I write 6 under poles, and carry 3 *R.* to the roods; the sum of these is 15 *R.*, which, divided by 4, gives 3 *A.* 3 *R.*; I write 3 under roods, and add 3 *A.* to the column of acres. Having completed the addition, the sum total is 139 *A.* 3 *R.* 6 *P.* 15 *Sq. Yds.* 5¼ *S. F.*

3. Required the sum of the four last numbers of the preceding example.

Ans. 125 *A.* 3 *R.* 9 *P.* 18 *Sq. Yds.* 8½ *S. F.*

£	s.	d.
11	11	9¾
78	16	2¼
89	12	11½
65	19	7¾
14	13	6¼
33	15	9¾
294	9	11¼

£	s.	d.
11	12	3½
32	14	7¾
59	17	8¼
43	10	2¾
86	4	11½
97	13	6¾

£	s.	d.
23	19	4½
31	6	11¼
49	18	7¾
63	13	10¼
77	12	9½
84	17	5¾
331	8	1

Cwt.	qr.	lb	oz.	dr.
13	2	18	12	8
29	1	23	14	9
34	3	27	15	11
67	3	19	12	15
45	1	25	14	7
23	3	26	11	6

A.	R.	P.	Yd.	S. F.
7	2	31	29	8
9	3	28	19	7
8	2	17	15	6
9	3	23	25	8
6	2	39	24	7
4	3	32	16	5
47	3	14	11	5

bar.	gal.	qt.
29	27	3
13	15	3
44	11	2
52	26	3
89	29	2
93	14	1

SUBTRACTION OF COMPOUND NUMBERS.

105. We place the less number under the greater in the same manner as for addition; and commencing with the lowest order, we subtract the units of each order in the less number from the corresponding units of the greater, writing each remainder under the order which gave it. But if the number of the units of any inferior order in the less number is greater than the number above it, we add as many units to the upper number as will make a unit of the next higher order, and having subtracted the lower number from the sum, we add a unit to the next higher order of the lower number. By this procedure, the numbers are both increased by the same quantity, which (art. 27,) does not affect their difference.

EXAMPLE 1.

	A.	*R.*	*P.*	*S. yds.*	*S. F.*
From	129	3	6	15	$5\frac{1}{4}$
we would take	125	3	9	18	$8\frac{1}{2}$
	3	3	36	26	8

Here, as I cannot subtract $\frac{1}{2}=\frac{2}{4}$ from $\frac{1}{4}$, I borrow a unit from the 5 square feet, which (art. 59) is equal to $\frac{4}{4}$; then $\frac{4}{4}+\frac{1}{4}=\frac{5}{4}$, and $\frac{5}{4}-\frac{2}{4}=\frac{3}{4}$, which I write underneath. Having borrowed a unit from the 5 square feet, instead of diminishing this by a unit, I add a unit to the 8, which makes 9; then, as $9>5$, I borrow a square yard; reducing this to square feet, and adding it to the 5, I have 14; then $14-9=5$, which I write under 8. Having borrowed a square yard, I add 1 to 18, which makes 19, and as $19>15$, I borrow a pole, which reduced to yards and added to 15 makes $45\frac{1}{4}$; subtracting 19, there remains $26\frac{1}{4}$, wherefore I write 26 underneath, and re-

ducing the $\frac{1}{4}$ of a square yard to square feet, I have $2\frac{1}{4}$ square feet; this being added to the $5\frac{3}{4}$ square feet already found, gives 8, which I leave under the column of feet. Having borrowed a pole, I add 1 to 9, which makes 10; then I say, not 10 from 6, but borrowing a rood or 40 poles, 10 from 46 leaves 36, which I write underneath. I then add 1 to 3, which makes 4, and borrowing an acre or 4 roods, I say 4 from 7 leaves 3, which I write underneath. Lastly, having borrowed an acre, I add 1 to 5, which makes 6, and 6 from 9 leaves 3, which being written underneath, completes the operation.

This example is a proof of the second example, (art. 104,) the number subtracted from being the sum total of the five numbers added; the number subtracted, the sum of the four last numbers, and the remainder the first number.

Prove the following examples by addition and subtraction.

2. From $\frac{1}{6}$ of $\frac{3}{4}$ of a pound sterling take $\frac{5}{7}$ of $\frac{7}{8}$ of a shilling. Ans. 1*s.* $10\frac{1}{2}$*d.*

3. From 3*T.* 0*cwt.* 2*qrs.* 5℔ 13*oz.* take $\frac{2}{3}$ of $\frac{5}{6}$ of a cwt. Ans. 2*T.* 19*cwt.* 3*qrs.* 27℔ 9*oz.* $7\frac{1}{9}$*drs.*

4. From 3*A.* 0*R.* 4*P.* 9*yds.* 3*S. F.* take $\frac{1}{3}$ of an acre. Ans. 2*A.* 2*R.* 30*P.* 29*yds.* 4*S. F.* 72*S. in.*

A.	*R.*	*P.*	*yd.*	*sq. ft.*
23	2	21	19	6
15	3	35	24	7
7	2	25	$24\frac{1}{4}$	8

£	*s.*	*d.*
29	6	$8\frac{1}{2}$
13	17	$9\frac{3}{4}$
15	8	$10\frac{3}{4}$

Cwt.	*qr.*	℔	*oz.*	*dr.*
22	1	13	7	8
11	3	17	8	11
10	1	23	14	13

bar.	*gal.*	*qt.*
13	19	2
7	29	3

OF THE MULTIPLICATION OF COMPOUND NUMBERS, AND WHAT IS CALLED PRACTICE.

106. We have seen (articles 98 and 100) that a compound number, when reduced either to its lowest denomination or to a fraction of the highest, may be operated upon in the same manner as an abstract number: the multiplication, therefore, of compound numbers, may always be reduced to the multiplication of a fraction by a fraction.

For example, if we demand the cost of 54 square yards 6 feet of work, at the rate of £9 4*s*. 6*d*. per square yard, it is evident that if we multiply the cost of one yard by the number of yards we shall have the cost of the whole. Now, reducing 6 feet (art. 99) to the fraction of a square yard, we have $\frac{2}{3}$, and (art. 60) reducing $54\frac{2}{3}$ square yards to an improper fraction, we have $\frac{164}{3}$ for the number of yards: also reducing 4*s*. 6*d*. to the fraction of a pound, we have $\frac{9}{40}$, and reducing £$9\frac{9}{40}$ to an improper fraction, we have $\frac{369}{40}$ for the pounds. Lastly, £$\frac{369}{40}$ being multiplied by $\frac{164}{3}$ gives £$\frac{5043}{10}$ or £504 6*s*. for the required cost.

This method applies to all compound numbers, but generally requires more calculation than that which we are about to explain.

107. When one of the given numbers is simple, either of them may be taken for the multiplicand or multiplier, yet care must be taken not to mistake the nature of the units in the product which is determined by the question.

For example, if we would find the value of 27 yards of cloth, at 3*s*. $9\frac{3}{4}$*d*. per yard, it is evident that the required value is equal to 3*s*. $9\frac{3}{4}$*d*. multiplied by 27.

Now, if we require the cost of the number of units in 27, at 3*s*. $9\frac{3}{4}$*d*. per unit, it is of no consequence to what species of quantity the number 27 *is applied*; we shall therefore consider the 27 as

shillings, and consequently we may either multiply 3*s*. 9¾*d*. by 27*s*. or 27*s*. by 3*s*. 9¾*d*., that is to say, either of the given numbers may be taken for the multiplicand or multiplier.

Therefore, we shall first multiply 3*s*. 9¾*d*. by 27, as follows:

	s.	*d*.	
	3	9¾	
		$5 \times 5 + 2 = 27$	
	19	0¾	5 times the multiplicand
		5	
4	15	3¾	5 times 5 or 25 times
	7	7½	twice the multiplicand
£5	2	11¼	25 times plus 2 or 27 times.

Having added twice the multiplicand to 25 times, we have 27 times, that is to say, we have £5 2*s*. 11¼*d*. for the required cost.

We will now take 27*s*. or £1 7*s*. for the multiplicand, and procede as follows:

		£	*s*.	
6*d*. is	½*s*.	1	7	
			3*s*.	9¾*d*.
		4	1	
3*d*.	½	0	13	6
½*d*.	⅙	0	6	9
¼*d*.	½	0	1	1½
		0	0	6¾
		5	2	11¼

I first multiply by 3*s*. and I have £4 1*s*. for the product; then, to multiply by 9¾*d*., I separate this into convenient aliquot parts, reasoning thus: as I have £1 7*s*. in the product for each unit in 3*s*., that is to say, for 1*s*., I should have half this product for 6*d*., because 6*d*. is ½*s*.; I therefore take the half of £1

7*s*., which gives 13*s*. 6*d*.; then for the 3 remaining pence in 9*d*. I say, as 6*d*. gave a product of 13*s*. 6*d*., I should have the half of this product for 3*d*. because 3*d*. is the half of 6*d*.; I therefore take the half of 13*s*. 6*d*., which gives 6*s*. 9*d*.: then, for the $\frac{3}{4}$*d*., separating this into $\frac{1}{2}$*d*. and $\frac{1}{4}$*d*., for the $\frac{1}{2}$*d*. I say, as 3*d*. gave a product of 6*s*. 9*d*. I should have $\frac{1}{6}$ of this product for $\frac{1}{2}$*d*., because $\frac{1}{2}$*d*. is $\frac{1}{6}$ of 3*d*.; I therefore take $\frac{1}{6}$ of 6*s*. 9*d*., which gives 1*s*. 1$\frac{1}{2}$*d*. Again, for the $\frac{1}{4}$*d*., as this is $\frac{1}{2}$ of $\frac{1}{2}$*d*. I take the half of 1*s*. 1$\frac{1}{2}$*d*., which gives 0*s*. 6$\frac{3}{4}$*d*. Lastly, adding the several products together, I have £5 2*s*. 11$\frac{1}{4}$*d*. as before.

Or, we may perform the same question thus:

	s.	£	*s*.	
3*d*. is	$\frac{1}{4}$	1	7	
			3*s*.	9$\frac{3}{4}$*d*.
		4	1	
			6	9
			6	9
$\frac{3}{4}$*d*.	$\frac{1}{4}$		6	9
			1	8$\frac{1}{4}$
		5	2	11$\frac{1}{4}$

Here, instead of separating the 9*d*. into 6*d*. and 3*d*., I separate it into 3*d*. plus 3*d*. plus 3*d*., and having found the product for 3*d*. I repeat it 3 times: then, as $\frac{3}{4}$*d*. is (art. 58) the same as $\frac{1}{4}$ of 3*d*. I take $\frac{1}{4}$ of 6*s*. 9*d*., the product given by 3*d*., and adding as before, I have the same result.

Hence the aliquot parts may be taken variously to suit the operator; however we should always endeavour to render the operation as simple, and to employ as few figures as possible. The last operation is, therefore, in both respects, preferable to either of the others.

This method of multiplying compound numbers, by aliquot parts, is much used by merchants, and is generally called practice.

Let the following examples be performed by the preceding method and by fractions.

Examples for Practice.

1. At 3*s*. 6*d*. what cost 187*yds*.?
   Ans. £32 14*s*. 6*d*.
2. At 4*s*. 8*d*. what cost 523*yds*.?
   Ans. £122 0*s*. 8*d*.
3. At 5*s*. $3\frac{3}{4}$*d*. what cost 315*yds*.?
   Ans. £83 13*s*. $5\frac{1}{4}$*d*.
4. At 19*s*. $1\frac{1}{2}$*d*. what cost 735*yds*.?
   Ans. £702 16*s*. $10\frac{1}{2}$*d*.
5. At 23*s*. $5\frac{3}{4}$*d*. what cost 1962*yds*.?
   Ans. £2303 6*s*. $1\frac{1}{2}$*d*.
6. At 22*s*. $11\frac{1}{4}$*d*. what cost 2627*yds*.?
   Ans. £3012 16*s*. $9\frac{3}{4}$*d*.

108. When both numbers are compound, we operate as in the following example:

What is the cost of 11*cwt*. 1*qr*. 14lb of sugar, at £3 15*s*. 6*d*. per cwt.?

	£	*s*.	*d*.	
1*qr*. is $\frac{1}{4}$*cwt*.	3	15	6	
5*s*. is $\frac{1}{4}$*l*.	11*cwt*.	1*qr*.	14lb	
	33			
	2	15		prod.
	2	15		for 15*s*.
6*d*. is $\frac{1}{10}$	2	15		
	0	5	6	
14lb is $\frac{1}{2}$*qr*.	0	18	$10\frac{1}{2}$	
	0	9	$5\frac{1}{4}$	
	42	18	$9\frac{3}{4}$	

I first multiply £3 15*s*. 6*d*. by 11, as in the preceding article, that is to say, having multiplied £3 by 11, I take parts of 11 considered as pounds for the 15*s*. 6*d*. Having thus taken 11 times £3 15*s*. 6*d*., which is the value of 1*cwt*., I have the value of 11*cwt*.

Then, to have the value of 1*qr.* I take ¼ of £3 15*s.* 6*d.*, the value of a cwt., which gives 18*s.* 10½*d.* Lastly, to have the value of 14lb, as this is the half of 1*qr.*, I take ½ of 18*s.* 10½*d.*, the value of a quarter, and having added the several products I have £42 18*s.* 9¾*d.* for the required cost.

The same may also be found thus:

	£	*s.*	*d.*	
1*qr.* is ¼*cwt.*	3	15	6	
				11*cwt.* 1*qr.* 14lb
	41	10	6	
14lb is ½*qr.*		18	10½	
		9	5¼	
	42	18	9¾	

I here multiply £3 15*s.* 6*d.* by 11, according to the first method exposed, (art. 107,) after which I take parts for the 1*qr.* 14lb. and having added, I have £42 18*s.* 9¾*d.* for the whole cost, as before.

Examples for Practice.

1. At £2 10*s.* 6*d.* what cost 4*cwt.* 3*qr.* 14lb? Ans. £12 6*s.* 2¼*d.*
2. At £3 14*s.* 8½*d.* what cost 7*cwt.* 0*qr.* 19lb? Ans. £26 15*s.* 7½*d.*+
3. At £2 3*s.* 9¾*d.* what cost 13*cwt.* 2*qr.* 7lb? Ans. £29 14*s.* 2½*d.*—
4. At £5 11*s.* 6¾*d.* what cost 9*cwt.* 1*qr.* 12lb? Ans. £52 3*s.* 10¾*d.*+
5. At £7 1*s.* 4½*d.* what cost 29*cwt.* 3*qr.* 17lb? Ans. £211 7*s.* 4¼*d.*+

109. The aliquot parts are frequently more complicate than those we have hitherto met with, in which case we sometimes introduce a false product, as in the following example:

Ex. 1. At the rate of £34 10*s.* 2*d.* per square yard, what is the cost of 17*S.* *yds.*?

	£	s.	d.
	34	10	2
	17		
	238		
	34		
1s. is $\frac{1}{10}$	8	10	
2d. is $\frac{1}{6}$	$\cancel{0}$	$\cancel{17}$	$\cancel{0}$
		2	10
	£586	12	10

Having multiplied £34 by 17, and also the 10*s.* by 17, in taking the half of £17 we have to multiply 2*d.*, which is $\frac{1}{6}$ of a shilling, and consequently $\frac{1}{6}$ of $\frac{1}{10}$ or $\frac{1}{60}$ of 10*s.*; but instead of taking $\frac{1}{60}$ of £8 10*s.*, the product given by 10*s.*, we make a false product by first taking the tenth of £8 10*s.*; this tenth, which is 17*s.*, is the product for 1*s.*: now, as 2*d.* is only the $\frac{1}{6}$ of a shilling, we bar this false product and write the sixth of it underneath; after which, adding as usual, we have £586 12*s.* 10*d.* for the required cost.

Ex. 2. How much work should we have for £34 10*s.* 2*d.*, at the rate of £1 for 17 yards?

It is evident that we should have 17 yards as often as the pound is contained in £34 10*s.* 2*d.*; we shall therefore multiply 17 yards by £34 10*s.* 2*d.*, as follows:

	S. yds.		
10*s.* is $\frac{1}{2}$*l.*	17		
	34*l.*	10*s.*	2*d.*
	68		
	51		
1*s.* is $\frac{1}{10}$	8	$4\frac{1}{2}$*S.F.*	
2*d.* is $\frac{1}{6}$	$\cancel{0}$	$\cancel{7}\frac{\cancel{13}}{\cancel{20}}$	
	0	$1\frac{11}{40}$	
	586*S.y.* $5\frac{31}{40}$*S.ft.*		

We first multiply 17*S. yds.* by 34; then, to multiply by 10*s.*, as this is the half of a pound, we take the half of 17*S. yds.*, which gives 8*S. yds.* $4\frac{1}{2}$*S. F.* To multiply by 2*d.* we first find the product for 1*s.* in taking the tenth of what we had for 10*s.*, that is to say, the tenth of 8*S. yds.* $4\frac{1}{2}$*S. F.*; this tenth is $7\frac{1}{2}\frac{3}{5}$ *S. F.*, but as it ought not to make a part of the product we bar it: then, as 2*d.* is $\frac{1}{6}$ of a shilling, we take $\frac{1}{6}$ of $7\frac{1}{2}\frac{3}{5}$*S. F.*; this sixth is $1\frac{1}{4}\frac{1}{6}$*S. F.*, which we write underneath, and having added the several products, the total product or quantity sought is 586*S. yds.* $5\frac{3}{4}\frac{1}{6}$*S. F.*

In this and the preceding example the factors are the same, but the two products are very different in the nature of their units. Hence we see the importance of distinguishing the nature of the units in the product, which is always made known by the conditions of the question.

Examples for Practice.

1. At £3 5*s.* 4*d.* what cost 185*yds.*?
 Ans. £604 6*s.* 8*d.*
2. At £2 9*s.* $3\frac{3}{4}$*d.* what cost 284*yds.*?
 Ans. £700 4*s.* 9*d.*
3. At £12 15*s.* 4*d.* what cost 316*T.*?
 Ans. £4034 5*s.* 4*d.*
4. At £13 12*s.* $0\frac{1}{2}$*d.* what cost 297*T.*?
 Ans. £4039 16*s.* $4\frac{1}{2}$*d.*
5. At £9 19*s.* $1\frac{1}{2}$*d.* what cost 921*T.*?
 Ans. £9169 14*s.* $1\frac{1}{2}$*d.*

OF THE DIVISION OF A COMPOUND NUMBER BY A SIMPLE NUMBER.

110. If the dividend and divisor represent quantities of different kinds, we first divide the highest order of units in the dividend by the divisor; we then reduce the remainder of this division to the next lower denomination, adding the units of the dividend

which are of this denomination; after which we divide, and procede with the remainder in the same manner as before. We continue thus to the lowest denomination, and collecting the several quotients, we have the total quotient or compound number sought.

EXAMPLE.

If 30 yards of cloth cost £10 9*s*. 4½*d*., what is that a yard?

```
      £    s.    d.
      10   9     4½
       20
      ____________
  30)209s. 4½d.(6s. 11¾d.
      180
      ___
       29s.
       12
      ___
  30)352(11d.
      330
      ___
       22d.
        4
       __
  30)90(3qrs.
      90
      __
      **
```

As 30 is not contained in 10, we reduce the £10 to shillings, to which adding the 9*s*. of the dividend, we have 209*s*.; we divide this by 30, which gives 6*s*. for the quotient and 29*s*. for the remainder; reducing this remainder to pence, and adding the 4*d*. of the dividend, we have 352*d*.; we divide this by 30, which gives 11*d*. for the quotient and 22*d*. for the remainder; reducing this remainder to farthings, and adding the two farthings of the dividend, we

have 90 farthings; we divide by 30, and have 3 farthings for the quotient, without remainder. Lastly, collecting the several quotients, we have 6*s.* 11¾*d.* for the price of a yard, as was required.

We may also perform the operation in dividing by the factors of 30, as follows:

	£	*s.*	*d.*
5)	10	9	4½
6)	2	1	10½
	*£*0	6	11¾

Examples for Practice.

1. If 185*yds.* cost *£*604 6*s.* 8*d.* what is that a yard? Ans. *£*3 5*s.* 4*d.*

2. If 284*yds.* cost *£*700 4*s.* 9*d.* what is that a yard? Ans. *£*2 9*s.* 3¾*d.*

3. If 316*T.* cost *£*4034 5*s.* 4*d.* what is that a ton? Ans. *£*12 15*s.* 4*d.*

4. If 297*T.* cost *£*4039 16*s.* 4½*d.* what is that a ton? Ans. *£*13 12*s.* 0½*d.*

5. If 921*T.* cost *£*9169 14*s.* 1½*d.* what is that a ton? Ans. *£*9 19*s.* 1½*d.*

111. When the dividend and divisor represent quantities of the same species, we must observe whether the quotient should or should not be of the same species with them, which the nature of the question will always decide. If the nature of the question requires that the quotient should be of the same kind, we procede as in the preceding article.

For example, suppose that with *£*17 we have gained *£*20 6*s.* 11¼*d.*, and that we would find the gain on each pound: it is evident that the quotient should be of the same kind as the divisor and dividend, that is to say, should be money, and that it *should* be the seventeenth part of *£*20 6*s.* 11¼*d.*; we therefore divide *£*20 6s. 11¼d. by 17, as in the

preceding article, which gives £1 3*s*. 11¼*d*. for the required gain.

But if the nature of the question requires that the quotient should be of a different kind, we reduce the divisor and dividend to the lowest denomination mentioned in the dividend; after which, considering the units of the dividend as being of the same kind with those of the quotient that we seek, we divide as in the preceding article.

For example, if we would know how much land can be bought for £314 19*s*. 6*d*., at the rate of £4 per acre, it is evident from the nature of the question that the quotient must be acres and parts of an acre, and that the number of these acres must be determined by the number of times that £4 is contained in £314 19*s*. 6*d*.: to simplify the numbers, therefore, we reduce the dividend to pence, which gives 75594; we also reduce the divisor to pence, which gives 960; we then divide 75594 considered as acres by 960, which gives 78*A*. 2*R*. 39*P*. for the quantity of land sought.

EXAMPLES.

1. How much corn can I buy for £26 10*s*., at the rate of 8*s*. a bushel? Ans. 66*bush*. 1*p*.
2. How much cloth may be bought for £20 16*s*. 4*d*., at 4*s*. a yard? Ans. 104*yds*. 0*qr*. 1⅓*n*.
3. Suppose that, having planted 25 bushels of potatoes, we have harvested 437*bush*. 2*p*., what is the rate of increase?

   Ans. 17*bush*. 2*p*. for each bushel, or 17½ for 1.

DIVISION OF A COMPOUND NUMBER BY A COMPOUND NUMBER.

112. When the divisor and dividend are both compound numbers, but of different kinds, we reduce the divisor to a fraction of the principal unit, and divide the dividend by this fraction reduced to

its lowest terms, that is to say, (art. 72,) we multiply the dividend by the denominator of this fraction, and divide the product by its numerator.

EXAMPLES.

1. If 15*cwt.* 2*qr.* 8lb of sugar cost £39 1*s.* 2*d.*, what is that a cwt.?

15*cwt.* 2*qr.* 8lb$=15\frac{4}{7}$*cwt.*$=\frac{109}{7}$*cwt.*

```
          £     s.    d.
         39     1     2
                      7
     ------------------
109)273     8     2(2l. 10s. 2d.
    218
    ---
     55
     20
    ---
109)1108(10
    1090
    ----
      18
      12
     ---
109)218(2
    218
    ---
    ***
```

I reduce 2 *qr.* 8lb to the fraction of a cwt., (art. 100,) which gives $\frac{4}{7}$*cwt.*; then, $15\frac{4}{7}$*cwt.* reduced to an improper fraction gives $\frac{109}{7}$*cwt.*

Now, to divide by $\frac{109}{7}$, we must multiply by 7 and divide by 109, (art. 72.) I therefore multiply £39 1*s.* 2*d.* by 7, and dividing the product £273 8*s.* 2*d.* by 109, I have £2 10*s.* 2*d.* for the answer.

2. If 13*yds.* 2*qrs.* of cloth cost £2 5*s.* $6\frac{3}{4}$*d.* what is that a yard? Ans. 3*s.* $4\frac{1}{2}$*d.*

3. If 14*cwt.* 3 *qrs.* 9lb of sugar cost £40 9*s.* 6*d.*, what is that a cwt.? Ans. £2 14*s.* 7*d.*+

4. If 15*T.* 16*cwt.* of hay cost £71 2*s.*, what is that a ton? Ans. £4 10*s.*

113. When the divisor and dividend are of the same kind, and the nature of the question requires that the quotient should also be of the same kind with them, we procede as in the preceding article; but if the question requires that the quotient should be of a different kind, we reduce the dividend and divisor to the lowest denomination mentioned in either; after which, considering the units of the dividend as being of the same kind as those of the quotient that we seek, we divide as usual.

Examples.

1. How much sugar may be bought for £39 1*s.* 2*d.*, at the rate of £2 10*s.* 2*d.* a cwt.?

Here the divisor and dividend are numbers of the same kind, but the nature of the question requires that the quotient should be of a different kind; we therefore reduce the dividend to pence, which gives 9374; we also reduce the divisor to pence, which gives 602; we then divide 9374 considered as hundred weights by 602, which gives 15*cwt.* 2*qrs.* 8℔ for the answer.

Examples for Practice.

1. At £3 5*s.* 4*d.* a yard, how many yards can I buy for £604 6*s.* 8*d.*? Ans. 185.

2. At £2 9*s.* 3¾*d.* a yard, how many yards can I buy for £700 4*s.* 9*d.*? Ans. 284.

3. At £12 15*s.* 4*d.* a ton, how many tons can I buy for £4034 5*s.* 4*d.*? Ans. 316.

4. At £13 12*s.* 0½*d.* a ton, how many tons can I buy for £4039 16*s.* 4½*d.*? Ans. 297.

5. At £9 19*s.* 1½*d.* a ton, how many tons can I buy for £9169 14*s.* 1½*d.*? Ans. 921.

6. At £4 10*s.* a ton, how much hay can I buy for £71 2*s.*? Ans. 15*T.* 16*cwt.*

QUESTIONS ON SECTION 12.

1. How do we add compound numbers?

2. How do we subtract one compound number from another?

3. How is the multiplication of two compound numbers reduced to the multiplication of a fraction by a fraction?

4. How do we multiply when one of the numbers is simple?

5. Is the taking of the aliquot parts subject to any rule, or are there various ways of doing this?

6. What is to be observed in taking these parts?

7. What do we observe with regard to the nature of the units in the product?

8. How do we procede when the given numbers are both compound?

9. How do we operate when the aliquot parts are complicate?

10. Does the nature of the units in the product depend upon the nature of those in either of the factors, and if not how is it determined? Give an example.

11. How is a compound number divided by a simple number?

12. What do we observe when the dividend and divisor represent quantities of the same species?

13. How is a compound number divided by a compound number of a different kind?

14. When the divisor and dividend are of the same kind, and the nature of the question requires that the quotient should also be of the same kind, how do we procede?

15. How do we operate when the nature of the question requires that the quotient should be of a different kind?

SECTION 13.

GEOMETRICAL RATIO AND SIMPLE PROPORTION.

114. When the first of two quantities of the same *kind is* divided by the second, the quotient is called

the *ratio* or *relation* of the first to the second. Thus the ratio of 8 to 4 is $\frac{8}{4}$ or 2; and the ratio of 4 to 8 is $\frac{4}{8}$ or $\frac{1}{2}$.

The ratio of two quantities is expressed by two points, one above the other, placed between them. Thus 6 : 3 signifies the ratio of 6 to 3, and is read, 6 *is to* 3.

The two quantities are called the *terms* of the ratio; the first term the *antecedent*, the second the *consequent*. Thus, in the ratio 6 : 3, six is the antecedent and three the consequent.

115. The value of a ratio is expressed by placing the antecedent over the consequent, in the form of a fraction, and the value of a fraction is not altered (art. 58) when both its terms are multiplied or divided by the same number. Therefore, *a ratio is not altered when both its terms are multiplied or divided by the same number.* Thus 24 : 32 is the same as 12 : 16, 6 : 8, or 3 : 4; that is to say, $\frac{24}{32}=\frac{12}{16}=\frac{6}{8}=\frac{3}{4}$.

116. A proportion is formed of two equal ratios. Thus 8 : 4 and 6 : 3 form a proportion, each ratio being equal to 2. Four points are placed between the ratios, thus, 8 : 4 : : 6 : 3, and the proportion is read, 8 *is to* 4 *as* 6 *is to* 3.

The first and last terms of a proportion are called the *extremes*, and the two middle terms the *means*.

From the above proportion we have $\frac{8}{4}=\frac{6}{3}$, and, multiplying each of these fractions by 4, the denominator of the first, we have $8=\frac{6\times 4}{3}$; again, multiplying each of these equals by 3, the denominator of the second, we have $8\times 3=6\times 4$. Hence *the product of the extremes is equal to the product of the means*, and this is evidently the case in every proportion, because the reasoning here applied will apply to any proportion whatever.

117. When four quantities are such that the product of the extremes is equal to the product of the means, they are in proportion.

For example, take any four such numbers as 3, 9, 2, 6; then, as $3\times6=9\times2$, if we divide each of these equals by 9, we have $\frac{3\times6}{9}=2$; again, if we divide each by 6, we have $\frac{3}{9}=\frac{2}{6}$; but $\frac{3}{9}$ is (art. 115) the same as the ratio 3 : 9, and $\frac{2}{6}$ the same as 2 : 6, and two equal ratios (art. 116) form a proportion. Wherefore the four numbers 3, 9, 2, 6 are proportionals, that is to say, 3 : 9 : : 2 : 6.

In a similar manner we might show that when the product of the extremes is not equal to the product of the means, the quantities are not in proportion.

Hence, if we put the extremes in the place of the means, and the means in the place of the extremes, or, if we change the places of the extremes or those of the means, the numbers will still be in proportion. Thus, from the above proportion, we have the following proportions:

3 : 9 : : 2 : 6
3 : 2 : : 9 : 6
6 : 2 : : 9 : 3
6 : 9 : : 2 : 3
9 : 3 : : 6 : 2
9 : 6 : : 3 : 2
2 : 3 : : 6 : 9
2 : 6 : : 3 : 9

and it is the same with every proportion, because in all these changes the product of the extremes is still equal to the product of the means.

118. The product of two numbers divided by either of them (art. 43) gives the other, and as the product of the extremes is the same as that of the means, *the product of the extremes divided by either mean gives the other mean, also the product of the means divided by either extreme gives the other extreme.* Therefore, when three terms of a proportion are given, we can easily find the fourth.

119. As the first term of a proportion has the same relation to the second that the third has to the fourth, if the first is greater than the second, the

third must be greater than the fourth; if the first is less than the second, the third must be less than the fourth. Hence, *if the fourth term is less than the third, the second must be less than the first; and if the fourth term is greater than the third, the second must be greater than the first.*

120. In trade many questions occur, from the nature of which three terms of a proportion are made known and the fourth is required. Two of the three given quantities are of the same kind, and the remaining one is of the same kind with the answer.

To solve a question of this kind, write the quantity which is of the same kind with the answer for the third term; then, from the reading of the question, find whether the answer or fourth term should be greater or less than the term written: if the answer should be greater, the greater of the two remaining terms must be the second; but if the answer should be less, the greater of the two remaining terms must be the first. See the preceding article.

Having arranged the three terms, divide the product of the means by the given extreme, which (art. 117) will give the extreme sought.

NOTE. The operation is much shortened by first dividing either of the means by the first term, or the first term by either mean; or, finally, by dividing either mean and the given extreme both by the same number, whenever this is practicable. Now it is evident that this may be done, first, because (art. 115) a ratio is not altered when both its terms are divided by the same number, and lastly, because the two means are to be multiplied together; and consequently it matters not, as to the result, which of them is taken for the consequent of the first ratio, only that there is an evident impropriety in saying that quantities of different kinds have a relation to one another. When some or all of the terms are compound numbers, care must be taken to reduce the terms of the first ratio to units of the same order, or to fractions of the same unit. The units of the

answer will be of the same kind as those to which the third term is reduced.

EXAMPLES.

1. If $5\frac{3}{4}$ yards of cloth cost \$19, what will $34\frac{1}{2}$ yards cost, at the same rate?

As the answer must be money, I place \$19 for the third term; then, (art. 118,) as $34\frac{1}{2}$*yds.* must evidently cost more than $5\frac{3}{4}$*yds.*, I place the greater of the two remaining terms second, and I have

$$\begin{array}{llll} yds. & yds. & \$ & \\ 5\frac{3}{4} : & 34\frac{1}{2} :: & 19 : & \text{the answer.} \end{array}$$

Or, reducing the mixed numbers to improper fractions,

$$\begin{array}{llll} yds. & yds. & \$ & \\ \frac{23}{4} : & \frac{69}{2} :: & 19 : & \text{the answer.} \end{array}$$

Then, to find the fourth proportional, I multiply the two means and divide the product by the given extreme. Now, to divide by a fraction, we invert it and multiply, (art. 72.) I therefore invert the first term, and have

$$\frac{\overset{2}{\cancel{4}} \times \overset{3}{\cancel{69}} \times 19}{\cancel{23} \times \cancel{2} \times 1} = 114 \text{ dollars}$$

for the answer.

2. If $34\frac{1}{2}$ yards of cloth cost \$114, what will $5\frac{3}{4}$ yards cost, at the same rate? Ans. \$19.

3. If $34\frac{1}{2}$ yards of cloth cost \$114, how much of the same can I buy for \$19? Ans. $5\frac{3}{4}$ yards.

4. If $5\frac{3}{4}$ yards of cloth cost \$19, how much of the same can I buy for \$114? Ans. $34\frac{1}{2}$ yards.

5. If $16\frac{2}{3}$ bushels of wheat cost $13\frac{8}{9}$ dollars, what will $120\frac{3}{4}$ bushels cost, at the same rate?

Ans. \$100,62$\frac{1}{2}$.

6. If $120\frac{3}{4}$ bushels of wheat cost \$100,62$\frac{1}{2}$, what will $16\frac{2}{3}$ bushels cost, at the same rate?

Ans. \$13,888$\frac{8}{9}$.

7. If $120\frac{3}{4}$ bushels of wheat cost \$100,62$\frac{1}{2}$, how much of the same can I buy for \$13,$\dot{8}$?

Ans. $16\frac{2}{3}$ bushels.

8. If $16\frac{2}{3}$ bushels of wheat cost $\$13\frac{8}{9}$, how much of the same can I buy for $100,625?

Ans. $120\frac{3}{4}$ bushels.

Make the same changes in each of the five succeding questions.

9. If 28℔ of tea cost £5 10*s*. 4*d*., what will 140℔ cost, at the same rate? Ans. £51 6*s*. 8*d*.

10. In what time will the interest of $625 equal the interest of $\$546,87\frac{1}{2}$ for 96 days?

Ans. 84 days.

11. Bought $\frac{5}{8}$ of a ship for $5670. What is $\frac{2}{3}$ of the ship worth, at that rate? Ans. $6048.

12. A person travelling 8 hours a day completes a journey in 14 days. In what time will he complete the same journey when he travels 12 hours a day at the same rate? Ans. 9*d*. 4*h*.

13. If a piece of land measuring 34*A*. 3*R*. 23*P*. cost £168 9*s*. 4*d*., how much will 56*A*. 2*R*. 13*P*. cost, at the same rate? Ans. £273 3*s*. $5\frac{3}{5}\frac{8}{5}\frac{3}{8}\frac{3}{3}d$.

14. Lent my neighbour $\$217\frac{1}{2}$ for 112 days. How long may I retain $870 of his money to be repaid?

Ans. 28 days.

BILLS OF PARCELS.

No. 1.

New-York, 11th March, 1826.

Messrs. Battelle, Collins, & Co.

Bo't. of Wetmore, McNab, & Co.

2315 yards cotton cloth, at $7\frac{2}{3}d$. per yard,
1007 yards Irish linen, at 1*s*, $7\frac{5}{8}d$. per do.
1421 yards checks, at $11\frac{7}{8}d$. per do.
1635 yards striped linen, at 3*s*. $5\frac{5}{12}d$. per do.
408 yards fine holland, at 3*s*. $7\frac{3}{8}d$. per do.
2175 yards sheeting, at 1*s*. $1\frac{5}{8}d$. per do.
1170 yards figured cloth, at 4*s*. $7\frac{5}{7}d$ per do.

£977 11 $6\frac{1}{4}$

No. 2.

New-York, 16th May, 1826.

Mr. Albert Riley,

Bo't. of Allen, Dunshee, & Co.

500 yards plain lawns, at 2*s*. 9$\frac{7}{8}$*d*. per yd.
640 yards striped do., at 2*s*. 3$\frac{1}{3}$*d*. per do.
858 yards flowered do., at 3*s*. 5$\frac{1}{8}$*d*. per do.
330 yards sprigged do., at 3*s*. 2$\frac{5}{6}$*d*. per do.
570 yards do. do., at 2*s*. 11$\frac{1}{6}$*d*. per do.
940 yards silk gauzes, at 3*s*. 1$\frac{3}{5}$*d*. per do.

£574 13 4+

QUESTIONS ON SECTION 13.

1. What is ratio, and how is it expressed?

2. What is the general name of the two quantities which form a ratio, and what the particular name of each?

3. Why is the value of a ratio still the same when both its terms are multiplied or divided by the same number?

4. What is proportion, and how is a proportion written?

5. What is the fundamental property of a proportion?

6. How is this easily ascertained?

7. What changes may be made in the position of the terms, so that they may still form a proportion?

8. When three terms of a proportion are given, how do we find the fourth?

9. When a question is proposed, from the nature of which three terms of a proportion are made known and the fourth required, how are the given terms arranged?

10. How may the operation sometimes be shortened?

11. How is the operation performed when some or all of the given terms are compound numbers?

THE END.

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